Burungale *Res Math Sci(2016)3:16*
DOI 10.1186/s40687-016-0060-2

Research in the Mathematical Sciences

RESEARCH

Open Access

# An $l \neq p$-interpolation of genuine $p$-adic L-functions

Ashay A. Burungale*

*Correspondence:
ashayburungale@gmail.com
Department of Mathematics,
University of Arizona,
617 N. Santa Rita Ave.,
P. O. Box 210089,
Tucson, AZ 85721, USA

**Abstract**

Let $\mathcal{F}$ be a totally real field, $l$ and $p$ distinct odd prime unramified in $\mathcal{F}$ and $\mathfrak{l}$ a prime above $l$. Let $\mathcal{K}/\mathcal{F}$ be a $p$-ordinary CM quadratic extension and $\lambda$ an arithmetic Hecke character over $\mathcal{K}$. Hida constructed a measure on the $\mathfrak{l}$-anticyclotomic class group of $\mathcal{K}$ interpolating the normalised Hecke $L$-values $L^{\mathrm{alg},\mathfrak{l}}(0, \lambda\nu)$, as $\nu$ varies over the finite order $\mathfrak{l}$-power conductor anticyclotomic characters. In this article, we interpolate the measures as $\lambda$ varies in a $p$-adic family. In particular, this gives $p$-adic deformation of the measures. An analogue holds in the case of self-dual Rankin–Selberg convolution of a Hilbert modular form and a theta series. In the case of root number $-1$, we describe an upcoming analogous interpolation of the $p$-adic Abel–Jacobi image of generalised Heegner cycles associated with the convolution.

**Keywords:** $p$-adic L-functions, Modular measure, Hilbert modular Shimura variety, Toric modular forms

**Mathematics Subject Classification:** Primary 11F33, 11F41, 11G18, 11R23

## Contents

## 1 Background

The variation of arithmetic invariants in a family seems to be a natural phenomena to explore. One expects the existence of a commutative $p$-adic L-function associated with a critical motive $\mathcal{M}$ over a tower of number fields with the Galois group being a commutative $p$-adic Lie group $\Gamma$, characterised by an interpolation of the $p$-stabilised critical $L$-values of $\mathcal{M}$ twisted by a dense subset of characters of $\Gamma$. One also expects that these $p$-adic L-functions can be interpolated as $\mathcal{M}$ itself varies in a $p$-adic family. In [6], Hida refers to the $p$-adic L-function along the family variable as the genuine $p$-adic L-function of the family. One thus expects $p$-adic interpolation of the genuine $p$-adic L-function of the family over the characters of $\Gamma$. When the family is self-dual with root number $-1$, the $L$-values identically vanish. In such a situation, the Bloch–Beilinson conjectures predict the existence of non-torsion cycles associated with the family which are homologically trivial. If a candidate for such cycles is available, one can investigate the existence of the interpolation for arithmetic invariants associated with the cycles, for example their $p$-adic Abel–Jacobi image.

This article concerns $l$-adic anticyclotomic Iwasawa theory of a $p$-ordinary CM field in a $p$-adic Hida family with $l \neq p$. Strictly speaking, we consider arithmetic Hecke characters over the CM field in a $p$-adic Hida family. A rich source of such characters arises from CM abelian varieties. In [7], Hida constructed an $l \neq p$-analogue of anticyclotomic $p$-adic L-function for arithmetic Hecke characters over a CM field. Let $\mathcal{F}$ be a totally real field and $\mathfrak{l}$ a prime above $l$. Let $\mathcal{K}/\mathcal{F}$ a CM quadratic extension and $\lambda$ an arithmetic Hecke character over $\mathcal{K}$. More precisely, Hida constructed a $\overline{\mathbf{Z}}_p$-valued measure $d\varphi_\lambda$ on the $\mathfrak{l}$-anticyclotomic Galois group $\Gamma_{\mathfrak{l}}^-$ over $\mathcal{K}$ interpolating the $\mathfrak{l}$-stabilised Hecke $L$-values $L^{\mathrm{alg},\mathfrak{l}}(0,\lambda\nu)$, as $\nu$ varies over the finite order $\mathfrak{l}$-power conductor characters of $\Gamma_{\mathfrak{l}}^-$. In this article, we interpolate the measures $d\varphi_\lambda$ as $\lambda$ varies in a $p$-adic family, namely in a $\Lambda$-adic family $\Psi$. This gives $p$-adic deformation of the measures. Strictly speaking, we construct a measure $d\varphi_\Psi$ interpolating a modification of $d\varphi_\lambda$ which interpolates $\mathfrak{l}p$-stabilised Hecke $L$-values $L^{\mathrm{alg},\mathfrak{l}p}(0,\lambda\nu)$. For a fixed $\nu$, the integral $\int_{\Gamma_{\mathfrak{l}}^-}\nu d\varphi_\Psi$ gives rise to the genuine $p$-adic L-function of the $\Lambda$-adic Hecke character $\Psi\nu$ in the sense of Hida introduced in [6]. We thus obtain a genuine $p$-adic L-function as a by-product of our $l \neq p$ consideration. Just to rephrase, the measure $d\varphi_\Psi$ gives an $l \neq p$-interpolation of the genuine $p$-adic L-function of the $\Lambda$-adic Hecke character $\Psi$ over the finite order characters of $\Gamma_{\mathfrak{l}}^-$. An analogue of the results holds in the case of self-dual Rankin–Selberg convolution of a Hilbert modular form and a theta series. In the case of root number $-1$ over the rationals, a candidate for the cycles alluded to above is generalised Heegner cycles. The construction is due to Bertolini–Darmon–Prasanna and generalises the one of classical Heegner cycles (cf. [1,2]). The cycle lives in a middle-dimensional Chow group of a fibre product of a Kuga–Sato variety arising from a modular curve and a self-product of a CM elliptic curve. In the case of weight two, the cycle coincides with a Heegner point and the $p$-adic Abel–Jacobi image with the $p$-adic formal group logarithm. In an ongoing work, we construct an analogous interpolation of the $p$-adic Abel–Jacobi image of the cycles. In this manner, the phenomena of variation seem to prevail even in self-dual situations with root number $-1$.

Let us introduce some notation. Let $\mathcal{F}$ be a totally real field of degree $d$ as above, $O$ the ring of integers and $p$ an odd prime unramified in $\mathcal{F}$. Let $h$ be the strict class number of $\mathcal{F}$. Fix two embeddings $\iota_\infty : \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let $\nu_p$ be the $p$-adic valuation induced by $\iota_p$ normalised so that $\nu_p(p) = 1$. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. Let $W(\mathbb{F})$ be the Witt ring and $\mathcal{W} = \iota_p^{-1}(W(\mathbb{F}))$. Let $l \neq p$ be an odd prime unramified in $\mathcal{F}$ and $\mathfrak{l}$ a prime above $l$. Let $\mathcal{K}$ be a CM quadratic extension of $\mathcal{F}$ as above. Let $\tau_{\mathcal{K}/\mathcal{F}}$ be the quadratic character associated with $\mathcal{K}/\mathcal{F}$ and $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ the relative different. Let $c$ denote the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(\mathcal{K}/\mathcal{F})$ via $\iota_\infty$. We assume the following hypothesis:

($ord$) Every prime of $\mathcal{F}$ above $p$ splits in $\mathcal{K}$.

The condition ($ord$) guarantees the existence of a $p$-adic CM type $\Sigma$ i.e. $\Sigma$ is a CM type of $\mathcal{K}$ such that the $p$-adic places induced by elements in $\Sigma$ via $\iota_p$ are disjoint from those induced by $\Sigma c$. We fix such a CM type. Let $\Omega_\infty = (\Omega_{\infty,\sigma})_\sigma \in (\mathbf{C}^\times)^\Sigma$ (resp. $\Omega_p = (\Omega_{p,\sigma})_\sigma \in (\overline{\mathbf{Z}}_p^\times)^\Sigma$) be the Archimedean (resp. $p$-adic) CM period of the CM abelian variety with CM type $(\mathcal{K}, \Sigma)$. For $a \in \mathbf{C}^\times$ and $\kappa \in \mathbf{Z}[\Sigma]$, let $a^\kappa = \prod_\sigma a^{\kappa_\sigma}$ and $\Omega^\kappa = \prod_\sigma \Omega_\sigma^{\kappa_\sigma}$.

Let $\lambda$ be an arithmetic Hecke character over $\mathcal{K}$ of infinity type $k\Sigma$ with $k \geq 0$ and conductor prime-to-$\mathfrak{l}$. For a non-negative integer $n$, let $\mathcal{K}_{\mathfrak{l}^n}^-$ be the anticyclotomic extension of $\mathcal{K}$ of conductor $\mathfrak{l}^n$ and $\Gamma_n^- = \mathrm{Gal}(\mathcal{K}_{\mathfrak{l}^n}^-/\mathcal{K})$. Let $\mathcal{K}_{\mathfrak{l}^\infty}^- = \bigcup_n \mathcal{K}_{\mathfrak{l}^n}^-$ i.e. the anticyclotomic extension of $\mathcal{K}$ of conductor $\mathfrak{l}^\infty$, $\Gamma_\mathfrak{l}^- = \mathrm{Gal}(\mathcal{K}_{\mathfrak{l}^\infty}^-/\mathcal{K}) = \varprojlim \Gamma_n^-$ and $\mathfrak{X}_\mathfrak{l}^-$ the set of finite order characters of $\Gamma_\mathfrak{l}^-$. From the results of Shimura and Katz, for $\nu \in \mathfrak{X}_\mathfrak{l}^-$ the normalised Hecke $L$-value

$$L^{\mathrm{alg},\mathfrak{l}p}(0, \lambda\nu) := \frac{\Gamma_\Sigma(k\Sigma)L^{(\mathfrak{l}p)}(0, \lambda\nu)}{\Omega_\infty^{k\Sigma}} \in \overline{\mathbf{Z}}_{(p)}. \tag{1.1}$$

Here $\Gamma_\Sigma(k\Sigma) = \prod_\sigma \Gamma(k)$ for the usual Gamma function $\Gamma(s)$ and $L^{(\mathfrak{l}p)}(s, \lambda\nu)$ is the (possibly) imprimitive $L$-function obtained from $L(s, \lambda\nu)$ by removing the Euler factors at all $v|p$ and $\mathfrak{l}$.

In [7], Hida constructed an $\overline{\mathbf{Z}}_p$-valued modular measure $d\varphi_\lambda$ on $\Gamma_\mathfrak{l}$ such that for $\nu \in \mathfrak{X}_\mathfrak{l}^-$, we have

$$\frac{1}{\Omega_p^{k\Sigma}} \cdot \int_{\Gamma_\mathfrak{l}^-} \nu d\varphi_\lambda \doteq L^{\mathrm{alg},\mathfrak{l}p}(0, \lambda\nu) \tag{1.2}$$

Here $\doteq$ denotes that the equality holds up to an explicit $p$-adic unit.

In this article, we consider the variation of $d\varphi_\lambda$ as $\lambda$ varies in a $p$-adic family. A rather systematic source of a $p$-adic family of Hecke characters is given by Hida's notion of a $\Lambda$-adic Hecke character. To briefly recall the notion, we introduce more notation. Let $\mathbf{\Gamma} = 1 + p\mathbf{Z}_p$ be the maximal torsion-free subgroup of $\mathbf{Z}_p^\times$ and $\gamma = (1 + p)$ a topological generator of $\mathbf{\Gamma}$. Let $W$ be a discrete valuation ring finite flat over $\mathbf{Z}_p$ and $\Lambda_W = W[[\mathbf{\Gamma}]]$. We identify $\Lambda_W = W[[T]]$ via $\gamma \leftrightarrow 1 + T$. We say that $P \in \mathrm{Spec}\,(\Lambda_W)(\overline{\mathbf{Q}}_p)$ is arithmetic of weight $k(P)$ with character $\epsilon_P : \mathbf{\Gamma} \to \mu_{p^\infty}(\overline{\mathbf{Q}}_p)$ if $P(1 + T - \epsilon_P(\gamma)\gamma^{k(P)-1}) = 0$ and $k(P) \geq 2$. Let $\mathbb{I}$ be a domain of finite rank over $\Lambda_W$. Let $Q$ be the quotient field of $\Lambda$, $\overline{Q}$

an algebraic closure and $\bar{\mathbb{I}}$ an integral closure of $\mathbb{I}$ in $\overline{Q}$. We say that $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ is arithmetic of weight $k(P)$ with character $\epsilon_P : \mathbf{\Gamma} \to \mu_{p^\infty}(\overline{\mathbf{Q}}_p)$ if it lies above an arithmetic (still denoted by) $P \in \mathrm{Spec}\,(\Lambda_W)(\overline{\mathbf{Q}}_p)$ of weight $k(P)$ and character $\epsilon_P$. An $\mathbb{I}$-adic Hecke character is a Galois character $\Psi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathcal{K}) \to \mathbb{I}^\times$ such that the arithmetic specialisation $\Psi_P$ is the $p$-adic avatar of an arithmetic Hecke character of infinity type $(k(P)-1)\Sigma$. For the notion of the prime-to-$p$ conductor of $\Psi$, we refer to Sect. 4.2. For $\nu \in \mathfrak{X}_{\mathfrak{l}}^{-}$, let $\mathbb{I}_\nu$ be the finite flat extension of $\mathbb{I}$ obtained by adjoining values of the finite order character $\nu$.

Our result regarding the variation is the following.

**Theorem A** *Let $\mathcal{F}$ be a totally real field and $p$ an odd prime unramified in $\mathcal{F}$. Let $l \neq p$ be an odd prime unramified in $\mathcal{F}$ and $\mathfrak{l}$ a prime above $l$. Let $\mathcal{K}/\mathcal{F}$ be a CM quadratic extension with a $p$-ordinary CM type $\Sigma$. Let $\Gamma_{\mathfrak{l}}^{-}$ be the $\mathfrak{l}$-anticyclotomic Galois group over $\mathcal{K}$. Let $\mathbb{I}$ be a domain of finite rank over $p$-adic one-variable Iwasawa algebra and $\bar{\mathbb{I}}$ the integral closure of $\mathbb{I}$ as above. Let $\Psi$ be an $\mathbb{I}$-adic Hecke character over $\mathcal{K}$ with prime-to-$p$ conductor prime-to-$\mathfrak{l}$. Then, there exists an $\bar{\mathbb{I}}$-valued measure $d\varphi_\Psi$ on $\Gamma_{\mathfrak{l}}^{-}$ such that*

$$\frac{1}{\Omega_p^{(k(P)-1)\Sigma}} \cdot \left(\int_{\Gamma_{\mathfrak{l}}^{-}} \nu \mathrm{d}\varphi_\Psi\right)_P = \frac{1}{\Omega_p^{(k(P)-1)\Sigma}} \cdot \int_{\Gamma_{\mathfrak{l}}^{-}} \nu \mathrm{d}\varphi_{\Psi_P} \doteq L^{\mathrm{alg},\mathfrak{l}p}(0, \Psi_P \nu). \tag{1.3}$$

Here $\nu$ is a finite order character of $\Gamma_{\mathfrak{l}}^{-}$, $P \in \mathrm{Spec}\,(\mathbb{I}_\nu)(\overline{\mathbf{Q}}_p)$ an arithmetic prime with weight $k(P)$, $\mathbb{I}_\nu$ the finite flat extension of $\mathbb{I}$ and CM periods $(\Omega_p, \Omega_\infty)$ as above. In particular, $\int_{\Gamma_{\mathfrak{l}}^{-}} \nu \mathrm{d}\varphi_\Psi \in \mathbb{I}_\nu$ equals the genuine $p$-adic L-function of the $\mathbb{I}_\nu$-adic Hecke character $\Psi\nu$.

In particular, we can consider $\mathrm{d}\varphi_\Psi$ as a $p$-adic deformation of modular measures constructed in [7] and [16]. For non-triviality of the measure $\mathrm{d}\varphi_\Psi$, we refer to Sect. 5.3. The non-triviality is based on the non-triviality results in [7] and [16].

We now give a sketch of the proof. Some of the notation used here is not followed in the rest of the article. In [7] and [16], the measure $\mathrm{d}\varphi_\lambda$ on $\Gamma_{\mathfrak{l}}^{-}$ is constructed via essentially a weighted sum of evaluation of a toric Eisenstein series $f_\lambda$ at the $\mathfrak{l}$-power conductor CM points on an underlying Hilbert modular Shimura variety. The Hecke $L$-values in consideration essentially equal such an evaluation. In these articles, the results are under the hypothesis that the conductor of $\lambda$ is prime-to-$p$. To remove it, we make an appropriate choice for the local sections corresponding to the toric Eisenstein series at the places dividing $p$. The latter is indeed technical novelty of the article and allows to treat the case of conductor being divisible by places above $p$. In consideration of $\Lambda$-adic family of Hecke characters, the conductor of the corresponding arithmetic Hecke characters is indeed divisible by places above $p$. As $P \in \mathrm{Spec}\,(\mathbb{I}_\nu)(\overline{\mathbf{Q}}_p)$ varies over arithmetic primes, we prove that the toric Eisenstein series $f_{\Psi_P}$ are interpolated by an $\mathbb{I}_\nu$-adic toric Eisenstein series $F_\Psi$. The $\mathbb{I}_\nu$-adic Eisenstein series is constructed based on an analysis of the $q$-expansion coefficients of the toric Eisenstein series. The $\Lambda$-adic measure $\mathrm{d}\varphi_\Psi$ is then constructed by a similar evaluation of $F_\Psi$ at CM points on an underlying $p$-ordinary Igusa tower as the measure $\mathrm{d}\varphi_\lambda$. Summarising, $p$-adic deformation of the toric Eisenstein series gives rise to the $p$-adic deformation of modular measures.

Let us briefly recall the history of "prime-to-$p$ Iwasawa theory". It first arose in Washington's work [26] as $l \neq p$-variation of Iwasawa's theorem on the class numbers of the $p$-adic cyclotomic towers of number fields. This result has an application to prove the non-vanishing of the corresponding Dirichlet $L$-values modulo $p$. In [23], Sinnott found a different way to prove this result based on the Zariski density of certain roots of unity on self-products of the multiplicative group $\mathbb{G}_m$ in characteristic $p$. In [7], Hida adapted this idea and generalised it to Hecke L-functions over CM fields based on the mod $p$ geometry of Hilbert modular Shimura varieties. Hida's work crucially relies on Chai's theory of Hecke-stable subvarieties of a mod $p$ Shimura variety. In [16], Hsieh refined computational aspects of Hida's strategy. For related results, we refer to Vatsal [25], Finis [4] and Sun [24].

The existence of genuine $p$-adic L-functions for an $\mathbb{I}$-adic Hecke character is proven in [5, §8]. Our by-product construction is slightly different and seems to give additional information, for example regarding the ring of definition. Moreover, our construction also works in the case of self-dual Rankin–Selberg convolution, and the by-product construction of the genuine $p$-adic L-function is perhaps new in this case. More precisely, an analogue of Theorem A holds for self-dual Rankin–Selberg convolution of a parallel weight $\Lambda$-adic Hilbert modular form and a parallel weight $\Lambda$-adic theta series. Based on the Waldspurger formula, the $L$-value in consideration is again essentially a sum of evaluation of a toric Hilbert modular form at the $\mathfrak{l}$-power conductor CM points (cf. [18]). The construction of the relevant $\bar{\mathbb{I}}$-valued measure is accordingly similar and we skip the details. In this article, we only consider $\Lambda$-adic Hecke character for the one-variable Iwasawa algebra $\Lambda$. We can also consider $\Lambda_{\mathcal{F}}$-adic Hecke character for the $(d+1)$-variable Iwasawa algebra $\Lambda_{\mathcal{F}}$. An analogue of Theorem A holds in this case as well.

Based on the $p$-adic Waldspurger formula in [1] and [2], we can show the existence of an analogous interpolation of the $p$-adic Abel–Jacobi image of the generalised Heegner cycles alluded to above. This is an ongoing work in progress. The construction is quite similar to that of the measure $d\varphi_{\Psi}$. We refer to Sect. 6 for a brief sketch in the case of weight two.

The $\mathbb{I}_{\nu}$-adic Eisenstein series would perhaps have other arithmetic applications. It essentially appears in [20] and is a key input to show a congruence between Katz $p$-adic L-functions over certain CM extensions of a fixed totally real field.

Finer questions about the non-triviality of the $\mathbb{I}$-adic measure seem to require new ideas (cf. the remark following Proposition 4.5). It would be interesting to see whether there is an Iwasawa main conjecture in the setting of the measure $d\varphi_{\Psi}$ i.e. as $\nu \in \mathfrak{X}_{\mathfrak{l}}^{-}$ varies. Also, it would be interesting to see whether the $l \neq p$-interpolation of genuine $p$-adic L-function of a $p$-adic family consisting the critical motive $\mathcal{M}$ exists in other contexts. We are tempted to believe the answer to be affirmative in the case of $p$-ordinary family of Hilbert modular forms.

The article is organised as follows. In Sect. 2, we recall certain generalities about Hilbert modular Shimura variety. In Sect. 3, we recall certain generalities about Hilbert modular forms. In Sect. 4, we describe anticyclotomic modular measures associated with a class of classical and $\Lambda$-adic Hilbert modular forms. This section is perhaps the technical heart of the article. In Sect. 4.1, we describe anticyclotomic modular measures associated with a class of classical Hilbert modular forms. In Sect. 4.2, we describe certain generalities about Hida's $\Lambda$-adic Hecke character. In Sect. 4.3, we describe anticyclotomic modular measures associated with a class of $\Lambda$-adic Hilbert modular forms. In Sect. 5, we prove Theorem A. In Sect. 5.1, we describe construction of a toric Eisenstein series following Hsieh. In Sect. 5.2, we describe a $\Lambda$-adic interpolation of the toric Eisenstein series in Sect. 5.1, thereby obtaining a $\Lambda$-adic toric Eisenstein series. In Sect. 5.3, we prove Theorem A. In Sect. 6, we briefly describe the upcoming analogous interpolation of the $p$-adic Abel–Jacobi image of generalised Heegner cycles associated with self-dual Rankin–Selberg convolution of an elliptic modular form and a theta series with root number $-1$.

*Notation* We use the following notation unless otherwise stated.

Let $\mathcal{F}_+$ denote the totally positive elements in $\mathcal{F}$. We sometime use $O$ to denote the ring of integers $\mathcal{O}_\mathcal{F}$. Let $\mathcal{D}_\mathcal{F}$ (resp. $D_\mathcal{F}$) be the different (resp. discriminant) of $\mathcal{F}/\mathbf{Q}$. Let $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ (resp. $D_{\mathcal{K}/\mathcal{F}}$) be the different (resp. discriminant) of $\mathcal{K}/\mathcal{F}$. Let $\mathbf{h}$ (resp. $\mathbf{h}_\mathcal{K}$) be the set of finite places of $\mathcal{F}$ (resp. $\mathcal{K}$). Let $v$ be a place of $\mathcal{F}$ and $w$ be a place of $\mathcal{K}$ above $v$. Let $\mathcal{F}_v$ be the completion of $\mathcal{F}$ at $v$, $\varpi_v$ an uniformiser and $\mathcal{K}_v = \mathcal{F}_v \otimes_\mathcal{F} \mathcal{K}$. For the $p$-ordinary CM type $\Sigma$ of $\mathcal{K}$ as above, let $\Sigma_p = \{w \in \mathbf{h}_\mathcal{K} | w|p \text{ and } w \text{ induced by } \iota_p \circ \sigma, \text{ for } \sigma \in \Sigma\}$.

For a number field $\mathcal{L}$, let $\mathbf{A}_\mathcal{L}$ be the adele ring, $\mathbf{A}_{\mathcal{L},f}$ the finite adeles and $\mathbf{A}_{\mathcal{L},f}^\square$ the finite adeles away from a finite set of places $\square$ of $\mathcal{L}$. For $a \in \mathcal{L}$, let $\mathfrak{il}_\mathcal{L}(a) = a(\mathcal{O}_\mathcal{L} \otimes_\mathbf{Z} \widehat{\mathbf{Z}}) \cap \mathcal{L}$. For a fractional ideal $\mathfrak{a}$, let $\widehat{\mathfrak{a}} = \mathfrak{a} \otimes \widehat{\mathbf{Z}}$. Let $G_\mathcal{L}$ be the absolute Galois group of $L$ and $\mathrm{rec}_\mathcal{L}$ : $\mathbf{A}_\mathcal{L}^\times \to G_\mathcal{L}^{ab}$ the geometrically normalised reciprocity law. Let $\psi_\mathbf{Q}$ be the standard additive character of $\mathbf{A}_\mathbf{Q}$ such that $\psi_\mathbf{Q}(x_\infty) = \exp(2\pi i x_\infty)$, for $x_\infty \in \mathbf{R}$. Let $\psi_\mathcal{L} : \mathbf{A}_\mathcal{L}/\mathcal{L} \to \mathbf{C}$ be given by $\psi_\mathcal{L}(y) = \psi_\mathbf{Q} \circ (\mathrm{Tr}_{\mathcal{L}/\mathbf{Q}}(y))$, for $y \in \mathbf{A}_\mathcal{L}$. We denote $\psi_\mathcal{F}$ by $\psi$.

## 2 Hilbert modular Shimura variety

In this section, we recall certain generalities about Hilbert modular Shimura variety.

In regard to the article, the section is preliminary. It briefly recalls the geometric theory of Hilbert modular Shimura variety, $p$-ordinary Igusa tower and CM points on them. The CM points inherently give rise to $p$-integral structure. The theory plays foundational role in the geometric theory of Hilbert modular forms and $p$-adic modular forms (cf. Sect. 3). Roughly speaking, classical modular forms (resp. $p$-adic modular forms) are functions on the Hilbert modular Shimura variety (resp. Igusa tower). The latter in turn play an underlying role in the construction of modular measures (cf. Sect. 4). Evaluation of $p$-adic modular forms at well chosen CM points is central to the construction.

### 2.1 Setup

In this subsection, we recall a basic setup regarding Hilbert modular Shimura variety. We refer to [7, §2.2] and [16, §2.1] for details.

Let the notation and hypotheses be as in the introduction. Let $G = \mathrm{Res}_{\mathcal{F}/\mathbf{Q}} GL_2$ and $h_0 : \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbb{G}_m \to G_{/\mathbf{R}}$ be the morphism of real group schemes arising from

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \mathbf{C}^\times$. Let $X$ be the set of $G(\mathbf{R})$-conjugacy classes of $h_0$. We have a canonical isomorphism $X \simeq (\mathbf{C} - \mathbf{R})^I$, where $I$ is the set of real places of $\mathcal{F}$. The pair $(G, X)$ satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(\mathrm{Sh}_K = \mathrm{Sh}_K(G, X))_K$ of quasi-projective smooth varieties over $\mathbf{Q}$ indexed by open compact subgroups $K$ of $G(\mathbf{A}_{\mathbf{Q},f})$. The pro-algebraic variety $\mathrm{Sh}_{/\mathbf{Q}}$ is the projective limit of these varieties. The complex points of these varieties are given as follows

$$\mathrm{Sh}_K(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathbf{Q},f})/K, \mathrm{Sh}(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathbf{Q},f})/\overline{Z(\mathbf{Q})}. \tag{2.1}$$

Here $\overline{Z(\mathbf{Q})}$ is the closure of the centre $Z(\mathbf{Q})$ in $G(\mathbf{A}_{\mathbf{Q},f})$ under the adelic topology.

Let us introduce some notation. Consider $V = \mathcal{F}^2$ as a two-dimensional vector space over $\mathcal{F}$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $\langle ., . \rangle : V \times V \to \mathcal{F}$ be the $\mathcal{F}$ bilinear pairing defined by $\langle e_1, e_2\rangle = 1$. Let $\mathcal{L} = Oe_1 \oplus O^* e_2$ be the standard $O$ lattice in V. For a fractional ideal $\mathfrak{b}$ of $O$, $\mathfrak{b}^* := \mathfrak{b}^{-1}\mathfrak{d}_{\mathcal{F}}^{-1}$. Here $\mathfrak{d}_?$ denotes the different of $?/\mathbf{Q}$, where ? equals $\mathcal{F}$ or $\mathcal{K}$. Sometimes, we denote $\mathfrak{d}_{\mathcal{F}}$ by $\mathfrak{d}$. For $g \in G(\mathbf{Q})$, $g' := \det(g)g^{-1}$. Note that $G(\mathbf{Q})$ has a natural right action on $\mathcal{F}^2$. For $x \in V$, consider the left action $gx := xg'$.

For a positive integer $N$ prime-to-$p$, let

$$U(N) = \left\{g \in G(\mathbf{A}_{\mathbf{Q},f}) \middle| g \equiv 1 \bmod N\mathcal{L}\right\}.$$

Let $h$ be the set of finite places of $\mathcal{F}$. For $v \in h$, let

$$K_v^0 = \left\{g \in GL_2(\mathcal{F}_v) | g(\mathcal{L} \otimes O_v) = \mathcal{L} \otimes O_v\right\}, \quad K_p^0 = \prod_{v|p} K_v^0.$$

From now on, we consider only the open compact subgroups $K$ of $G(\mathbf{A}_{\mathbf{Q},f})$ for which $K_p$ equals $K_p^0$ and $U(N) \subset K$, for some prime-to-$p$ integer $N$. Let

$$K_0(\mathfrak{l}) = \left\{g \in K | e_2 g \in O^* e_2 \bmod \mathfrak{l}\mathcal{L}\right\}.$$

For a non-negative integer $n$, let

$$K_1^n = \left\{g \in K \middle| g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod p^n\right\}.$$

### 2.2 *p*-Integral model

In this subsection, we briefly recall generalities about a canonical $p$-integral smooth model of a Hilbert modular Shimura variety. The description gives rise to the $p$-integral structure on the space of Hilbert modular forms. We refer to [8, §4.2] for details.

Let the notation and hypotheses be as in the introduction and Sect. 2.1. Hilbert modular Shimura variety $Sh_{/\mathbf{Q}}$ represents a functor classifying abelian schemes having multiplication by $O$ along with additional structure, where $O$ is the ring of integers of $\mathcal{F}$ (cf. [8, §4.2] and [21]). When the prime $p$ is unramified in $\mathcal{F}$, a $p$-integral interpretation of the functor leads to a $p$-integral smooth model of $Sh/G(\mathbf{Z}_p)_{/\mathbf{Q}}$.

The $p$-integral interpretation is given by

$$\begin{aligned} &\mathcal{G}^{(p)} : SCH_{/\mathbf{Z}_{(p)}} \to SETS \\ &S \mapsto \left\{(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S}\right\} / \sim . \end{aligned} \tag{2.2}$$

Here

(PM1) $A$ is abelian scheme over $S$ of dimension of $d$.

(PM2) $\iota : O \hookrightarrow \mathrm{End}_S A$ is an algebra embedding.

(PM3) $\bar{\lambda}$ is the polarisation class of a homogeneous polarisation $\lambda$ of degree prime-to-$p$ up to scalar multiplication by $\iota(O_{(p),+}^{\times})$, where $O_{(p),+} := \{a \in O_{(p)} | \sigma(a) > 0, \forall \sigma \in I\}$. Let $^t$ denote the Rosati involution induced by the polarisation class on $\mathrm{End}_S A \otimes \mathbf{Z}_{(p)}$. We then have $\iota(l)^t = \iota(l)$ for $l \in O$. Let us fix an isomorphism $\zeta : \mathbf{A}_{\mathbf{Q},f} \simeq \mathbf{A}_{\mathbf{Q},f}(1)$ with the Tate twist. We can thus regard the Weil pairing $e^{\lambda}$ induced by $\lambda$ as an $\mathcal{F}$-alternate form

$$e^{\lambda} : V^{(p)}(A) \times V^{(p)}(A) \to O^* \otimes_{\mathbf{Z}} \mathbf{A}_{\mathbf{Q},f}^{(p)}.$$

Here $V^{(p)}(A) = \mathcal{T}^{(p)}(A) \otimes \mathbf{Q}$, where $\mathcal{T}^{(p)}(A)$ is as in (PM4). Let $e^{\eta}$ denote the $\mathcal{F}$-alternate form $e^{\eta}(x, x') := \langle x\eta, x'\eta\rangle$. Then,

$$e^{\lambda} = ue^{\eta}$$

for some $u \in \mathbf{A}_{\mathcal{F},f}^{(p)}$.

(PM4) Let $\mathcal{T}^{(p)}(A)$ be the prime-to-$p$ Tate module $\varprojlim_{p\nmid N} A[N]$. $\eta^{(p)}$ is a prime-to-$p$ level structure given by an $O$-linear isomorphism $\eta^{(p)}: O^2 \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(p)} \simeq \mathcal{T}^{(p)}(A)$, where $\widehat{\mathbf{Z}}^{(p)} = \prod_{l\neq p} \mathbf{Z}_l$.

(PM5) Let $\mathrm{Lie}_S(A)$ be the relative Lie algebra of $A$. There exists an $O \otimes_{\mathbf{Z}} \mathcal{O}_S$-module isomorphism

$$\mathrm{Lie}_S(A) \simeq O \otimes_{\mathbf{Z}} \mathcal{O}_S$$

locally under the Zariski topology of $S$.

The notation $\sim$ denotes up to a prime-to-$p$ isogeny.

**Theorem 2.1** (Kottwitz) *The functor $\mathcal{G}^{(p)}$ is represented by a smooth pro-algebraic scheme $Sh^{(p)}_{/\mathbf{Z}_{(p)}}$. Moreover, there exists an isomorphism given by*

$$Sh^{(p)} \times \mathbf{Q} \simeq Sh/G(\mathbf{Z}_p)_{/\mathbf{Q}}.$$

*(cf. [8, § 4.2.1]).*

Let $g \in G(\mathbf{A}^{(p)}_{\mathbf{Q},f})$ act on $Sh^{(p)}_{/\mathbf{Z}_{(p)}}$ via

$$x = (A, \iota, \bar{\lambda}, \eta^{(p)}) \mapsto gx = (A, \iota, \bar{\lambda}, \eta^{(p)} \circ g). \tag{2.3}$$

For $K$ neat, we have the corresponding level-$K^{(p)}$ Shimura variety $Sh^{(p)}_{K/\mathbf{Z}_{(p)}}$. Regarding the former condition, we only remark that a sufficiently small $K$ is always neat. For the moduli interpretation, we consider the functor $\mathcal{G}^{(p)}_K$ essentially satisfying the above hypotheses with the level $K^{(p)}$-structure instead of the hypotheses (PM4) (cf. [8, §4]).

Let $\mathfrak{c}$ be an ideal of $O$ prime-to-$p$. Let $\mathbf{c} \in (\mathbf{A}^{(p)}_{\mathcal{F},f})^{\times}$ such that $\mathfrak{c} = \mathfrak{il}_{\mathcal{F}}(\mathbf{c})$. We say that $x \in Sh^{(p)}_K(S)$ is $\mathfrak{c}$-polarised, if there exists $\lambda \in \bar{\lambda}$ [cf. (PM3)] such that for $u$ as in (PM4), we have $u \in \mathbf{c}\det(K)$. We consider the sub functor $\mathcal{G}^{(p)}_{\mathfrak{c},K}$ of the $\mathfrak{c}$-polarised quadruples. This functor is represented by a geometrically irreducible scheme $\mathrm{Sh}^{(p)}_{K/\mathbf{Z}_{(p)}}$, and we have

$$\mathrm{Sh}^{(p)}_K = \bigsqcup_{[\mathfrak{c}]\in \mathrm{Cl}^+_{\mathcal{F}}(K)} \mathrm{Sh}^{(p)}_K(\mathfrak{c}). \tag{2.4}$$

Here $\mathrm{Cl}^+_{\mathcal{F}}(K)$ is the narrow ray class group of $\mathcal{F}$ of level $\det(K)$.

### 2.3 Igusa tower

In this subsection, we briefly recall the notion of $p$-ordinary Igusa tower over the $p$-integral Hilbert modular Shimura variety. The Igusa tower underlies the geometric theory of $p$-adic Hilbert modular forms. We refer to [8, Ch. 8] for details.

Let the notation and hypotheses be as in the introduction and Sect. 2.2. In particular, $\mathcal{W}$ is the strict henselisation inside $\overline{\mathbf{Q}}$ of the local ring of $\mathbf{Z}_{(p)}$ corresponding to $\iota_p$, $W(\mathbb{F})$ is the Witt ring, and $\mathbb{F}$ is the residue field of $\mathcal{W}$.

Let $\mathrm{Sh}^{(p)}_{/\mathcal{W}} = \mathrm{Sh}^{(p)}(G, X) \times_{\mathbf{Z}_{(p)}} \mathcal{W}$ and $\mathrm{Sh}^{(p)}_{/\mathbb{F}} = Sh^{(p)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.

From now, let Sh (resp. $\mathrm{Sh}_K$) denote $\mathrm{Sh}^{(p)}_{/W}$ or $\mathrm{Sh}^{(p)}_{/\mathbb{F}}$ (resp. $\mathrm{Sh}^{(p)}_{K/W}$ or $\mathrm{Sh}^{(p)}_{K/\mathbb{F}}$). The base will be clear from the context. Let $\mathcal{A}$ be the universal abelian scheme over Sh.

Let $\mathrm{Sh}^{\mathrm{ord}}$ be the subscheme of Sh on which the Hasse invariant does not vanish. It is an open dense subscheme. Over $\mathrm{Sh}^{\mathrm{ord}}$, the connected part $\mathcal{A}[p^m]^{\circ}$ of $\mathcal{A}[p^m]$ is étale locally isomorphic to $\mu_{p^m} \otimes_{\mathbf{Z}_p} O^*$ as an $O_p$-module for $O_p = O \otimes \mathbf{Z}_p$.

We now define the Igusa tower. For $m \in \mathbb{N}$, the $m$th layer of the Igusa tower over $\mathrm{Sh}^{\mathrm{ord}}$ is defined by

$$Ig_m = \underline{\mathrm{Isom}}_{O_p}(\mu_{p^m} \otimes_{\mathbf{Z}_p} O^*, \mathcal{A}[p^m]^\circ). \tag{2.5}$$

Note that the projection $\pi_m : Ig_m \to \mathrm{Sh}^{\mathrm{ord}}$ is finite and étale. The full Igusa tower over $\mathrm{Sh}^{\mathrm{ord}}$ is defined by

$$Ig = Ig_\infty = \varprojlim Ig_m = \underline{\mathrm{Isom}}_{O_p}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} O^*, \mathcal{A}[p^\infty]^\circ). \tag{2.6}$$

($\acute{E}t$) Note that the projection $\pi : Ig \to \mathrm{Sh}^{\mathrm{ord}}$ is étale.

Let $x$ be a closed ordinary point in Sh. We have the following description of the level $p^\infty$-structure on $A_x[p^\infty]$.

(PL) Let $\eta_p^\circ$ be a level $p^\infty$-structure on $A_x[p^\infty]^\circ$. For the primes $\mathfrak{p}$ in $O$ dividing $p$, it is a collection of level $\mathfrak{p}^\infty$-structures $\eta_{\mathfrak{p}}^\circ$, given by isomorphisms $\eta_{\mathfrak{p}}^\circ : O_{\mathfrak{p}}^* \simeq A_x[\mathfrak{p}^\infty]^\circ$, where $O_{\mathfrak{p}}^* = O^* \otimes O_{\mathfrak{p}}$. The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism $\eta_{\mathfrak{p}}^{\acute{e}t} : O_{\mathfrak{p}} \simeq A_x[\mathfrak{p}^\infty]^{\acute{e}t}$. Thus, we get a level $p^\infty$-structure $\eta_p^{\acute{e}t}$ on $A_x[p^\infty]^{\acute{e}t}$ from $\eta_p^\circ$.

Let V be an irreducible component of Sh and $V^{\mathrm{ord}}$ denote $V \cap \mathrm{Sh}^{\mathrm{can}}$. Let $I$ be the inverse image of $V^{\mathrm{ord}}$ under $\pi$. In [8, Ch.8] and [11], it has been shown that

(Ir) $I$ is an irreducible component of $Ig$.

For a prime-to-$p$ ideal $\mathfrak{c}$ of $O$ and $K$ as above, we can analogously define the $p$-ordinary Igusa tower $Ig_K(\mathfrak{c})$ over level-$K^{(p)}$ Shimura variety $Sh_K^{(p)}(\mathfrak{c})$.

### 2.4 CM points

In this subsection, we briefly recall some notation regarding CM points on the Igusa tower. The notion is originally due to Shimura and underlies the theory of Shimura varieties. We refer to [22] for details.

Let the notation and hypotheses be as in the introduction and Sect. 2.3. In particular, $\mathcal{K}/\mathcal{F}$ is a $p$-ordinary CM quadratic extension. Let $\mathfrak{a} \subset \mathcal{O}_{\mathcal{K}}$ be an $O$-lattice. Let $R(\mathfrak{a}) = \{\alpha \in \mathcal{O}_{\mathcal{K}} | \alpha\mathfrak{a} \subset \mathfrak{a}\}$ be the corresponding order of $\mathcal{K}$ and $\mathfrak{f}(\mathfrak{a}) \subset O$ the corresponding conductor ideal. Recall that $R(\mathfrak{a}) = O + \mathfrak{f}(\mathfrak{a})\mathcal{O}_{\mathcal{K}}$.

Let $\Sigma$ be a $p$-ordinary CM type as in the introduction. By CM theory of Shimura–Taniyama–Weil (cf. [22]), the complex torus $X(\mathfrak{a})(\mathbf{C}) = \mathbf{C}^\Sigma/\Sigma(\mathfrak{a})$ is algebraisable to a CM abelian variety of CM type $(\mathcal{K}, \Sigma)$. Here $\Sigma(\mathfrak{a}) = \left\{\iota_\infty(\sigma(a))_\sigma | a \in \mathfrak{a}\right\}$. When the conductor $\mathfrak{f}(\mathfrak{a})$ is prime-to-$p$, the abelian variety $X(\mathfrak{a})$ extends to an abelian scheme over $\mathcal{W}$. We denote it by $X(\mathfrak{a})_{/\mathcal{W}}$.

In this case, a construction of a quadruple $(\iota_{\mathfrak{a}}, \overline{\lambda}_{\mathfrak{a}}, \eta^{(p)}(\mathfrak{a}), \eta_p^{\mathrm{ord}}(\mathfrak{a}))_{/W}$ satisfying (PM2)–(PM4) and (PL) is given in [16, §3]. By definition, this gives rise to a CM point

$$x(\mathfrak{a}) = \left(X(\mathfrak{a}), \iota_{\mathfrak{a}}, \overline{\lambda}_{\mathfrak{a}}, \eta^{(p)}(\mathfrak{a}), \eta_p^{\mathrm{ord}}(\mathfrak{a})\right) \in Ig(W).$$

### 2.5 Tate objects

In this subsection, we briefly recall some notation regarding Tate objects on the Hilbert modular Shimura variety. They naturally arise during the construction of toroidal compactifications of the Hilbert modular Shimura variety. In turn, they give rise to the key notion of $q$-expansion of Hilbert modular forms. We refer to [19, § 1.1] and [8, § 4.1.5] for details.

Let the notation and hypotheses be as in the introduction and Sect. 2.3. Let $\mathfrak{L}$ be a set of $d$ linearly independent elements $l \in \mathrm{Hom}(\mathcal{F}, \mathbf{Q})$ such that $l(\mathcal{F}_+) > 0$. Let $L$ be a lattice in $\mathcal{F}$, $n$ a positive integer, $L_{\mathfrak{L},n} = \{x \in L | l(x) > -n, \forall l \in \mathfrak{L}\}$ and $A((L, \mathfrak{L})) = \varinjlim A[\![L_{\mathfrak{L},n}]\!]$. Pick two fractional ideals $\mathfrak{a}, \mathfrak{b}$ of $O$ prime-to-$p$. To this pair, Mumford associated a certain abelian variety with real multiplication $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)_{/\mathbf{Z}((\mathfrak{ab},\mathfrak{L}))}$ endowed with a canonical $O$-action $\iota_{\mathrm{can}}$. Formally, $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) = \mathfrak{a}^* \otimes_{\mathbf{Z}} \mathbb{G}_m/q^{\mathfrak{b}}$. It is also endowed with a canonical polarisation $\lambda$, prime-to-$p$ level structure $\eta^{(p)}_{\mathrm{can}}$, $p^\infty$-level structure $\eta^{\mathrm{ord}}_{p,\mathrm{can}}$ and a generator $\omega_{\mathrm{can}}$ of $\Omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)}$. We thus obtain a Tate object

$$x(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)) = \left(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \iota_{\mathrm{can}}, \lambda_{\mathrm{can}}, \eta^{(p)}_{\mathrm{can}}, \eta^{\mathrm{ord}}_{p,\mathrm{can}}\right).$$

## 3 Hilbert modular forms

In this section, we recall certain generalities about geometric theory of Hilbert modular forms. We also briefly recall the adelic notion. The former plays a key role in the construction of modular measures in Sect. 4, while the latter in the construction of toric Eisenstein series in Sect. 5.

### 3.1 Classical Hilbert modular forms

In this subsection, we recall the notion of classical Hilbert modular forms. We refer to [8, §4.2] and [16, §2.5] for details.

#### *3.1.1 Geometric definition*

We first recall the geometric definition of classical Hilbert modular forms of parallel weight.

Let the notation and assumptions be as in Sect. 2. Let $k$ be a positive integer and $R$ a $\mathbf{Z}_{(p)}$-algebra. Let $T$ be the torus $\mathrm{Res}_{O/\mathbf{Z}}\mathbb{G}_m$ and $\underline{k}$ the character of $T$ arising from $t \mapsto N_{\mathcal{F}/\mathbf{Q}}(t)^k$, where $t \in O^\times$ and $N_{\mathcal{F}/\mathbf{Q}}$ the norm map. A classical Hilbert modular form of weight $k$ and level $K$ over $R$ is a function $f$ of isomorphism classes of $x = (\underline{\mathcal{A}}, \omega)$ where $\underline{\mathcal{A}} \in \mathrm{Sh}^{(p)}_K(S)$ and $\omega$ is a differential form generating $H^0(A, \Omega_{A/S})$ over $O \otimes_{\mathbf{Z}} S$ for an $R$-algebra $S$, locally under the Zariski topology such that the following conditions are satisfied.

- (Gc1) If $x \simeq x'$, then $f(x) = f(x') \in S$.
- (Gc2) $f(x \otimes S') = \rho(f(x))$ for any $R$-algebra homomorphism $\rho : S \to S'$.
- (Gc3) $f(\underline{\mathcal{A}}, s\omega) = \underline{k}(s)^{-1} f(\underline{\mathcal{A}}, \omega)$, where $s \in T(S)$.
- (Gc4) $f|_{\mathfrak{a},\mathfrak{b}}(q) := f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \iota_{\mathrm{can}}, \lambda_{\mathrm{can}}, \eta^{(p)}_{\mathrm{can}}, \omega_{\mathrm{can}}) \in R[\![q^{\mathfrak{ab}_{\geq 0}}]\!]$, where $\mathfrak{ab}_{\geq 0} = (\mathfrak{ab} \cap \mathcal{F}_+) \cup \{0\}$. We say that $f|_{\mathfrak{a},\mathfrak{b}}(q)$ is the $q$-expansion of $f$ at the cusp $(\mathfrak{a}, \mathfrak{b})$.

Let $M_k(K, R)$ be the space of the functions satisfying the above conditions (Gc1–4). For $f \in M_k(K, R)$, we have the following fundamental $q$-expansion principle (cf.[8, Thm. 4.21]).

($q$-exp) The $q$-expansion map $f \mapsto f|_{\mathfrak{a},\mathfrak{b}}(q) \in R[\![q^{\mathfrak{ab}_{\geq 0}}]\!]$ determines $f$ uniquely.

#### *3.1.2 Adelic definition*

We now briefly describe relation of the geometric definition of Hilbert modular forms to the adelic counterpart.

Let the notation and assumptions be as in Sect. 3.1.1. In particular, $k$ is a positive integer.

For $\tau = (\tau_\sigma)_{\sigma\in\Sigma}$ and $g = \left(\begin{bmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{bmatrix}\right)_{\sigma\in\Sigma} \in G(\mathbf{R})$, let

$$J(g,\tau)^k = \prod_{\sigma\in\Sigma}(c_\sigma\tau_\sigma + d_\sigma)^k.$$

Let $f$ be a Hilbert modular form of weight $k$ over $\mathbf{C}$. In view of the complex uniformisation of the Hilbert modular variety in (2.1), we may regard $f$ as a function $f : X^+ \times G(\mathbf{A}_{\mathbf{Q},f}) \to \mathbf{C}$ (also see [16, §2.5.2]). Let $\varphi : G(\mathbf{A}_{\mathbf{Q},f}) \to \mathbf{C}$ be the function given by

$$f(\tau, g_f) = \varphi(g)\cdot J(g_\infty, \mathbf{i})^k(\det g_\infty)^{-k\Sigma}|\det g|_{\mathbf{A}}^{k/2}.$$

Here $\mathbf{i} = (i)_{\sigma\in\Sigma}$, $g_\infty \in G(\mathbf{R})$ such that $g_\infty\mathbf{i} = \tau$ and $\det g_\infty > 0$. Moreover, $g = (g_\infty, g_f)$ and $|\cdot|_{\mathbf{A}}$ is the adelic norm. It turns out that $\varphi$ is an adelic Hilbert modular form.

Via the above identity, we can also start with an adelic Hilbert modular form and obtain a classical Hilbert modular form. Representation theoretic methods can often be used to construct adelic Hilbert modular forms. In Sect. 5, we adopt this approach to construct toric Eisenstein series.

### 3.2 *p*-Adic Hilbert modular forms

In this subsection, we recall the geometric definition of $p$-adic Hilbert modular forms. The theory is due to Katz and Hida. We refer to [8, §8] for details.

Let the notation and assumptions be as in Sect. 2. Let $R$ be a $p$-adic algebra. A $p$-adic Hilbert modular form of level $K$ over $R$ is a function $f$ of isomorphism classes of $x = (\underline{\mathcal{A}}, \eta_p^{\mathrm{ord}}) \in Ig_K(S)$ defined over any $p$-adic $R$-algebra $S$ such that the following conditions are satisfied.

(Gp1) If $x \simeq x'$, then $f(x) = f(x') \in S$.

(Gp2) $f(x \otimes S') = \rho(f(x))$ for any $p$-adic $R$-algebra homomorphism $\rho : S \to S'$.

(Gp3) $f|_{\mathfrak{a},\mathfrak{b}}(q) := f(x(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q))) \in R[\![q^{\mathfrak{ab}_{\geq 0}}]\!]$. We say that $f|_{\mathfrak{a},\mathfrak{b}}(q)$ is the $q$-expansion of $f$ at the cusp $(\mathfrak{a},\mathfrak{b})$.

Let $V(K,R)$ be the space of the functions satisfying the above conditions (Gp1-3). We can regard a classical Hilbert modular form as a $p$-adic Hilbert modular form as follows. Let $f \in M_k(K,R)$ and $(\underline{\mathcal{A}}, \eta_p^{\mathrm{ord}}) \in Ig_K(S)$. Note that $\eta_p^{\mathrm{ord}}$ induces an isomorphism $\eta_p^{\mathrm{ord}} : O^* \otimes_{\mathbf{Z}_p} \mathrm{Lie}(\widehat{\mathbb{G}}_m) \to \mathrm{Lie}(A)$. It follows that $(\eta_p^{\mathrm{ord}})^*(\frac{\mathrm{d}t}{t})$ generates $H^0(A,\Omega_A)$ as an $O \otimes_{\mathbf{Z}} R$-module. We regard $f$ as a $p$-adic Hilbert modular form of level $K$ via

$$f \mapsto \hat{f}\left(\underline{\mathcal{A}}, \eta_p^{\mathrm{ord}}\right) := f\left(\underline{\mathcal{A}}, \left(\eta_p^{\mathrm{ord}}\right)^*\left(\frac{\mathrm{d}t}{t}\right)\right).$$

Moreover, $\hat{f}|_{\mathfrak{a},\mathfrak{b}}(q) = f|_{\mathfrak{a},\mathfrak{b}}(q)$ and we have an embedding $M_k(K,R) \hookrightarrow V(K,R)$.

For $f \in V(K,R)$, we have the following fundamental $q$-expansion principle (cf. [8, Thm. 4.21])

($q$-exp)' The $q$-expansion map $f \mapsto f|_{\mathfrak{a},\mathfrak{b}}(q) \in R[\![q^{\mathfrak{ab}_{\geq 0}}]\!]$ determines $f$ uniquely.

### 3.3 Hecke operators

In this subsection, we recall the definition of certain Hecke operators on the space of $p$-adic Hilbert modular forms.

Let the notation and assumptions be as in Sect. 2. For $K$ as in Sect. 2.2 and $g \in G(\mathbf{A}_{\mathbf{Q},f}^{(p)})$, let $gK = gKg^{-1}$. Note that the action $x \mapsto gx$ (cf. (2.3)) gives rise to an isomorphism $\mathrm{Sh}_K \simeq \mathrm{Sh}_{gK}$. For $f \in V(K,R)$, let $f|g$ be given by

$$(f|g)(x) = f(gx). \tag{3.1}$$

It follows that $f|g \in V(gK, R)$.

Now suppose that $f \in V(K_0(\mathfrak{l}), R)$. Let $f|U_{\mathfrak{l}}$ be given by

$$f|U_{\mathfrak{l}} = \sum_{u \in O^*/\mathfrak{l}O^*} f\left| \begin{bmatrix} \varpi_{\mathfrak{l}} & u \\ 0 & 1 \end{bmatrix} \right., \tag{3.2}$$

where $\varpi_{\mathfrak{l}}$ is an uniformiser of $O_{\mathfrak{l}}$. It follows that $f|U_{\mathfrak{l}} \in V(K_0(\mathfrak{l}), R)$. We thus obtain an operator $U_{\mathfrak{l}}$ on $V(K_0(\mathfrak{l}), R)$. It plays a key role in the construction of the modular measures in Sect. 4.

### 3.4 $\Lambda$-Adic Hilbert modular forms

In this subsection, we recall the definition of $\Lambda$-adic Hilbert modular forms of parallel weight due to Hida. We refer to [9] for details.

Let the notation and hypotheses be as in the introduction and Sect. 2. Let $\mathfrak{N}$ be a prime-to-$p$ ideal of $O$ and $Cl_{\mathcal{F}}(\mathfrak{N}p^{\infty})$ the strict ray class group modulo $\mathfrak{N}p^{\infty}$. Let $N : Cl_{\mathcal{F}}(\mathfrak{N}p^{\infty}) \to \mathbf{Z}_p^{\times}$ be the norm map arising from $\mathfrak{a} \mapsto |O/\mathfrak{a}|$, where $\mathfrak{a}$ is a prime-to-$\mathfrak{N}p$ ideal. Let $\Gamma \simeq 1 + p\mathbf{Z}_p$ be the maximal torsion-free quotient of $\mathbf{Z}_p^{\times}$. Via the projection, we have $\langle N \rangle : Cl_{\mathcal{F}}(\mathfrak{N}p^{\infty}) \to \Gamma$. Let $\Delta = \ker(\langle N \rangle)$ and $\mathbf{\Gamma} = \mathrm{Im}(\langle N \rangle)$. As $\mathbf{\Gamma} \simeq \mathbf{Z}_p$, we have a splitting $Cl_{\mathcal{F}}(\mathfrak{N}p^{\infty}) = \Delta \times \mathbf{\Gamma}$.

Let $\gamma$ be a topological generator of $\mathbf{\Gamma}$. Let $\Lambda = \mathbf{Z}_p[\![\mathbf{\Gamma}]\!]$. We identify $\Lambda$ with the Iwasawa algebra $\mathbf{Z}_p[\![T]\!]$, via $\gamma \leftrightarrow 1 + T$. For a valuation ring $W$ finite flat over $\mathbf{Z}_p$, let $\Lambda_W = \Lambda \otimes_{\mathbf{Z}_p} W$. Let $\mathbb{I}$ be a domain of finite rank over $\Lambda_W$. We say that $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ is arithmetic of weight $k(P)$ with character $\epsilon : \mathbf{\Gamma} \to \mu_{p^\infty}(\mathbf{C}_p)$ if $k(P) \geq 2$ and $P(1 + T - \epsilon(\gamma)\gamma^{k(P)-1}) = 0$.

Let $F$ be a formal power series given by

$$F = \sum_{\beta \in \mathfrak{a}\mathfrak{b}_{\geq 0}} a_\beta(F) q^\beta \in \mathbb{I}[\![q^{\mathfrak{a}\mathfrak{b}_{\geq 0}}]\!]. \tag{3.3}$$

**Definition 3.1** (Hida) We say that $F$ is an $\mathbb{I}$-adic Hilbert modular form of level $K$ if for every arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ of weight $k(P)$ and character $\epsilon_P$, the formal power series

$$F_P = \sum_{\beta \in \mathfrak{a}\mathfrak{b}_{\geq 0}} P(a_\beta(F)) q^\beta \in \overline{\mathbf{Q}}_p[\![q^{\mathfrak{a}\mathfrak{b}_{\geq 0}}]\!] \tag{3.4}$$

is the $q$-expansion $f_{P,\mathfrak{a},\mathfrak{b}}(q)$ of a classical Hilbert modular form $f_P \in M_k(K_1^n, \overline{\mathbf{Q}}_p)$. Here $n$ is the $p$-exponent of $\mathrm{Im}(\epsilon)$.

Let $M_\Lambda(K, \mathbb{I})$ be the space of $\mathbb{I}$-adic Hilbert modular forms of level $K$. We later give an example of an $\mathbb{I}$-adic toric Eisenstein series (cf. Sect. 5.2).

An $\mathbb{I}$-adic Hilbert modular form of level $K$ is in fact a $p$-adic Hilbert modular form of level $K$ over $\mathbb{I}$ (cf. [9, §4.4.1]). Thus, it has functorial interpretation as in Sect. 3.2 and satisfies (Gp1)–(Gp3). In particular, the Hecke operators as in Sect. 3.3 are defined also for $\mathbb{I}$-adic Hilbert modular forms.

Let $\kappa : O_p^{\times} \to \Lambda^{\times}$ be the universal cyclotomic character given by $u \mapsto \langle N_p(u) \rangle \in \mathbf{\Gamma} \subset \Lambda^{\times}$, where $u \in O_p^{\times}$.

An $\mathbb{I}$-adic Hilbert modular form also satisfies the following (cf. [9]):

(Gp$\Lambda$) $F(\underline{\mathcal{A}}, \eta_p^{\mathrm{ord}} \circ u) = \kappa(u) F(\underline{\mathcal{A}}, \eta_p^{\mathrm{ord}})$, where $(\underline{\mathcal{A}}, \eta_p^{\mathrm{ord}}) \in Ig(S)$, $S$ an $\mathbb{I}$-algebra as in Sect. 3.2 and $u \in O_p^{\times}$.

## 4 Deformation of modular measures

In this section, we describe anticyclotomic measures associated with a class of classical and $\Lambda$-adic Hilbert modular forms.

The construction of such measures associated with classical Hilbert modular forms is due to Hida and Hsieh. The heart of this section is the construction of $p$-adic deformation of these measures. The construction builds in the geometric framework of Sects. 2 and 3.

### 4.1 Modular measures

In this subsection, we firstly describe an anticyclotomic measure associated with a class of classical Hilbert modular form due to Hida. The construction is perhaps formal. We then discuss its non-triviality also due to Hida. The non-triviality is quite subtle and relies on Chai's theory of Hecke-stable subvarieties of a mod $p$ Shimura variety.

Let the notation and hypotheses be as in Sects. 1–3. Recall that $\mathcal{K}_{\mathfrak{l}^n}^-$ is the anticyclotomic extension of $\mathcal{K}$ of conductor $\mathfrak{l}^n$ and $\Gamma_n^- = \mathrm{Gal}(\mathcal{K}_{\mathfrak{l}^n}/\mathcal{K})$. Let $R_n = O + \mathfrak{l}^n\mathcal{O}_{\mathcal{K}}$. Note that $\mathrm{Pic}(R_n) = \Gamma_n^-$. Let

$$U_n = (\mathbf{C}_1)^{\Sigma} \times (\widehat{R}_n) \subset (\mathbf{C}^{\times})^{\Sigma} \times \mathbf{A}_{\mathcal{K},f}^{\times}$$

be a compact subgroup for $\mathbf{C}_1$ is the unit complex circle. Via the reciprocity map, we identify

$$\Gamma_n^- = \mathcal{K}^{\times}\mathbf{A}_{\mathcal{F}}^{\times}\backslash\mathbf{A}_{\mathcal{K}}^{\times}/U_n.$$

Let $[\cdot]_n : \mathbf{A}_{\mathcal{K}}^{\times} \to \Gamma_n^-$ be the quotient map. For $a \in \mathbf{A}_{\mathcal{K}}^{\times}$, let $x_n(a)$ be the CM point $x([a]_n)$ associated with the ideal class $[a]_n$ defined in Sect. 2.4. Also, let $[a] = \varprojlim_n [a]_n \in \Gamma_{\mathfrak{l}}^-$.

Let $\lambda$ be an arithmetic Hecke character as in the introduction and $\widehat{\lambda}$ its $p$-adic avatar. Let $f \in V(K_0(\mathfrak{l}), \mathcal{O})$, for a finite flat extension $\mathcal{O}$ over $\mathbf{Z}_p$ such that the following conditions are satisfied:

(MC1) $f$ is an $U_{\mathfrak{l}}$-eigenform with the eigenvalue $a_{\mathfrak{l}}(f) \in \overline{\mathbf{Z}}_p^{\times}$ and
(MC2) $f(x_n(ta)) = \widehat{\lambda}(a)^{-1}f(x_n(t))$ for $a \in U_n\mathbf{A}_{\mathcal{F}}^{\times}$.

Following Hida, let $\mathrm{d}\varphi_f$ be the measure on $\Gamma_{\mathfrak{l}}^-$ such that for $\phi : \Gamma_n^- \to \overline{\mathbf{Z}}_p$, we have

$$\int_{\Gamma_{\mathfrak{l}}^-} \phi \mathrm{d}\varphi_f = (a_{\mathfrak{l}}(f))^{-n} \sum_{[t]_n \in \Gamma_n^-} f(x_n(t))\widehat{\lambda}(t)\phi([t]_n). \tag{4.1}$$

In view of the condition (MC2), the expression on the right-hand side is well defined if we change $t$ to $ta$, for $a \in U_n\mathcal{F}^{\times}$. The measure is well defined in view of the condition (MC1), (3.2) and the fact that

$$\left(x_n(t) \middle| \begin{bmatrix} \varpi_{\mathfrak{l}} & u \\ 0 & 1 \end{bmatrix}\right)_{u \in O^*/\mathfrak{l}O^*}$$

are all the CM points associated with $\pi_{n+1,n}^{-1}([t]_n)$ for the projection $\pi_{n+1,n} : \Gamma_{n+1}^- \twoheadrightarrow \Gamma_n^-$ (cf. [7, §3.1 & §3.4]).

We now suppose that $f$ is a classical Hilbert modular defined over a number field. In [7], Hida treats the case of nearly holomorphic Hilbert modular forms. We restrict to the classical case as it suffices for the later applications.

To discuss non-triviality of the above measure, we introduce more notation. Let $\Delta_{\mathfrak{l}}$ be the torsion subgroup of $\Gamma_{\mathfrak{l}}^-$, $\Gamma_{\mathfrak{l}}^{\mathrm{alg}}$ the subgroup of $\Gamma_{\mathfrak{l}}^-$ generated by $[a]$ for $a \in (\mathbf{A}_{\mathcal{K}}^{(\mathfrak{l}p)})^{\times}$ and $\Delta_{\mathfrak{l}}^{\mathrm{alg}} = \Gamma_{\mathfrak{l}}^{\mathrm{alg}} \cap \Delta_{\mathfrak{l}}$. Let $\mathcal{B}$ be a set of representatives of $\Delta_{\mathfrak{l}}/\Delta_{\mathfrak{l}}^{\mathrm{alg}}$ and $\mathcal{R}$ a set of representatives of $\Delta_{\mathfrak{l}}^{\mathrm{alg}}$ in $(\mathbf{A}_{\mathcal{K}}^{(\mathfrak{l}p)})^{\times}$. Let $\rho : \mathbf{A}_{\mathcal{K}}^{\times} \hookrightarrow \mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$ be the toric embedding and $\varsigma \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K}}^{(\mathfrak{l}p)})$ defined in [16, §3.1 and §3.2], respectively. For $a \in (\mathbf{A}_{\mathcal{K}}^{(\mathfrak{l}p)})^{\times}$, let

$$f|[a] = f|\rho_{\varsigma}(a), \rho_{\varsigma}(a) = \varsigma^{-1}\rho(a)\varsigma. \tag{4.2}$$

Also, let

$$f^{\mathcal{R}} = \sum_{r \in \mathcal{R}} \widehat{\lambda}(r) f|[r]. \tag{4.3}$$

We consider the following hypothesis.

(H) Let $l^{r(\lambda)}$ be the order of the $l$-Sylow subgroup of $\mathbb{F}_p[\lambda]^{\times}$. There exists a strict ideal class $\mathfrak{c} \in \mathrm{Cl}_{\mathcal{F}}$ such that $\mathfrak{c} = \mathfrak{c}(\mathfrak{a})$ for some $R$-ideal $\mathfrak{a}$ and for every $u \in O$ prime-to-$\mathfrak{l}$, there exists $\beta \equiv u \bmod \mathfrak{l}^{r(\lambda)}$ such that $\mathbf{a}_{\beta}(f^{\mathcal{R}}, \mathfrak{c}) \in \overline{\mathbf{Z}}_p^{\times}$. Here $\mathbf{a}_{\beta}(f^{\mathcal{R}}, \mathfrak{c})$ is the $\beta$th Fourier coefficient of $f^{\mathcal{R}}$ at the cusp $(O, \mathfrak{c}^{-1})$.

We have the following non-triviality of the modular measure.

**Theorem 4.1** *(Hida) In addition to (ord), suppose that the hypothesis (H) holds. Then, we have*

$$\int_{\Gamma_{\mathfrak{l}}^{-}} \nu \mathrm{d}\varphi_f \in \overline{\mathbf{Z}}_p^{\times},$$

*for "almost all" $\nu \in \mathfrak{X}_{\mathfrak{l}}^{-}$ (cf. [7, Thm. 3.2 and Thm 3.3]).*

Here "almost all" means except for all but finitely many $\chi \in \mathfrak{X}_{\mathfrak{l}}^{-}$ if $\deg \mathfrak{l} = 1$ and a Zariski dense subset of $\mathfrak{X}_{\mathfrak{l}}^{-}$, otherwise.

### 4.2 Λ-Adic Hecke characters

In this subsection, we describe certain generalities about Hida's notion of $\Lambda$-adic Hecke character.

Let the notation and hypotheses be as in Sects. 1–3. In particular, $\mathcal{K}/\mathcal{F}$ is a CM quadratic extension. In what follows, we consider Hecke characters over $\mathcal{K}$. We recall that $\mathbb{I}$ is a domain of finite rank over the Iwasawa algebra $\Lambda_W$, where $W$ is discrete valuation ring finite flat over $\mathbf{Z}_p$.

**Definition 4.2** (Hida) An $\mathbb{I}$-adic Hecke character is a continuous Galois character $\Psi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathcal{K}) \to \mathbb{I}^{\times}$ such that for every arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, the arithmetic specialisation $\Psi_P$ is the $p$-adic avatar of an arithmetic Hecke character of infinity type $(k(P)-1)\Sigma$.

We say that $\Psi$ is self-dual if for every arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, the arithmetic specialisation $\Psi_P$ is self-dual (cf. [16, §1]). This is also equivalent to the existence of an arithmetic prime $P_0 \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ such that the arithmetic specialisation $\Psi_{P_0}$ is self-dual. We similarly define $\Psi$ being not residually self-dual.

Let $\widehat{\mathcal{O}}_{\mathcal{K}}^{(p)} = \prod_{l \neq p}(\mathcal{O}_{\mathcal{K}} \otimes \mathbf{Z}_l)$. Note that the restriction $\Psi : (\widehat{\mathcal{O}}_{\mathcal{K}}^{(p)})^{\times} \to \mathbb{I}^{\times}$ is a finite order character as $\mathbb{I}^{\times}$ is an almost $p$-profinite group. Thus, there exists an integral ideal $\mathfrak{C}^{(p)}(\Psi)$ maximal among ideals $\mathfrak{a}$ prime-to-$p$ such that $(1 + \mathfrak{a}\widehat{\mathcal{O}}_{\mathcal{K}}^{(p)}) \cap (\widehat{\mathcal{O}}_{\mathcal{K}}^{(p)})^{\times} \subset \ker \Psi$. We say that $\mathfrak{C}^{(p)}(\Psi)$ is the prime-to-$p$ conductor of $\Psi$.

**Lemma 4.3** *Let $\Psi$ be an $\mathbb{I}$-adic Hecke character. Then, the prime-to-p conductor $\mathfrak{C}^{(p)}(\Psi)$ of $\Psi$ equals the prime-to-p conductor of the arithmetic specialisation $\Psi_P$ for any $P \in$ Spec $(\mathbb{I})(\overline{\mathbf{Q}}_p)$. (cf. [14, Prop. 3.2])*

We now recall construction of an $\mathbb{I}$-adic Hecke character due to Hida (cf. [15, §1.4]). The notion naturally arises in the context of $p$-ordinary family of CM modular forms.

Recall that $\Sigma$ is a $p$-ordinary CM type of the CM quadratic extension $\mathcal{K}/\mathcal{F}$. For $\mathfrak{p}|p$ in $\mathcal{F}$, let $\mathfrak{p} = \mathfrak{P}\mathfrak{P}^c$ for $\mathfrak{P} \in \Sigma_p$. Let $\lambda$ be an arithmetic Hecke character of conductor $\mathfrak{C}$ with infinity type $k\Sigma$ such that $k > 1$ and the conductor $\mathfrak{C}$ is outside $\Sigma_p^c$. Let $\mathfrak{C}^{(p)}$ be the prime-to-$p$ part of $\mathfrak{C}$. Let $\mathfrak{N}$ be the prime-to-$p$ part of $N_{\mathcal{K}/\mathcal{F}}(\mathfrak{C})\mathcal{D}_{\mathcal{K}/\mathcal{F}}$, where $N_{\mathcal{K}/\mathcal{F}}$ is the norm map and $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ the discriminant of the extension $\mathcal{K}/\mathcal{F}$. Let $\lambda^-$ be the Hecke character given by $\lambda^-(\mathfrak{a}) = \lambda(\mathfrak{a}^c\mathfrak{a}^{-1})$. Let $\mathfrak{C}(\lambda^-)$ be the conductor of $\lambda^-$.

Let $\widehat{\lambda}$ be the $p$-adic avatar of $\lambda$. We regard $\widehat{\lambda}$ as a character of the ray class group $\mathrm{Cl}_{\mathcal{K}}(\mathfrak{C}^{(p)}\mathfrak{S}^\infty)$ of $\mathcal{K}$ of conductor $\mathfrak{C}^{(p)}\mathfrak{S}^\infty$, where $\mathfrak{S} = \prod_{\mathfrak{P}\in\Sigma_p}\mathfrak{P}$. Let

$$\widehat{\lambda}_0(x) = \exp_p\left(\frac{1}{k-1}\log_p(\widehat{\lambda}(x))\right),$$

where $\exp_p$ (resp. $\log_p$) is the $p$-adic exponential (resp. $p$-adic Iwasawa logarithm). Note that $\widehat{\lambda}_0$ is the $p$-adic avatar of an arithmetic Hecke character of conductor $\mathfrak{C}\mathfrak{S}^e$ with infinity type $\Sigma$ for some integer $e \geq 0$.

Let $\mathcal{O}_\Sigma^\times = \varprojlim_n(\mathcal{O}_\mathcal{K}/\mathfrak{S}^n)^\times = O_p^\times$. We have the following commutative diagram:

$$\begin{tikzcd}
O_p^\times/O^\times \arrow[r] \arrow[d, "\langle N\rangle"] & \mathcal{O}_\Sigma^\times/\mathcal{O}_\mathcal{K}^\times \arrow[r] \arrow[d, "\langle\Phi\rangle"] & \mathrm{Cl}_\mathcal{K}(\mathfrak{C}^{(p)}\mathfrak{S}^\infty) \arrow[d, "\widehat{\lambda}_0"] \\
\mathbf{\Gamma} \arrow[r, "="] & \mathbf{\Gamma} \arrow[r, "\subset"] & \mathbf{\Gamma}_\mathcal{K}.
\end{tikzcd}$$

Here $\mathbf{\Gamma}_\mathcal{K} = \mathrm{Im}(\widehat{\lambda}_0) \subset \mathbf{C}_p^\times$ isomorphic to $\mathbf{Z}_p$ as a topological group and $\langle\Phi\rangle(x) = \langle\prod_{\sigma\in\Sigma}\sigma(x)\rangle \in \mathbf{\Gamma}$. As $\mathbf{\Gamma}_\mathcal{K}$ is $\mathbf{Z}_p$-free of rank 1, there exists a decomposition $\mathrm{Cl}_\mathcal{K}^-(\mathfrak{C}(\lambda^-)p^\infty) = \Delta_\mathcal{K} \times \mathbf{\Gamma}_\mathcal{K}$ for $\Delta_\mathcal{K} = \ker(\widehat{\lambda}_0)$ compatible with the decomposition $\mathrm{Cl}_\mathcal{F}(\mathfrak{N}p^\infty) = \Delta \times \mathbf{\Gamma}$ (cf. Sect.3.4) such that $\Delta$ maps inside $\Delta_\mathcal{K}$. We also note that $\mathbf{\Gamma}$ is an open subgroup of $\mathbf{\Gamma}_\mathcal{K}$. For a finite flat valuation ring $W$ over $\mathbf{Z}_p$, we thus conclude that $W[\![\mathbf{\Gamma}_\mathcal{K}]\!]$ is regular domain finite flat over $\Lambda_W$. We say that $P \in \mathrm{Spec}\,(W[\![\mathbf{\Gamma}_\mathcal{K}]\!])(\overline{\mathbf{Q}}_p)$ is arithmetic if it is above an arithmetic prime of $\mathrm{Spec}\,(\Lambda_W)(\overline{\mathbf{Q}}_p)$. Let $\upsilon$ be the character given by the composition:

$$\upsilon : \mathrm{Cl}_\mathcal{K}(\mathfrak{C}^{(p)}\mathfrak{S}^\infty) \xrightarrow{\widehat{\lambda}_0} \mathbf{\Gamma}_\mathcal{K} \hookrightarrow W[\![\mathbf{\Gamma}_\mathcal{K}]\!]^\times.$$

For an arithmetic $P \in \mathrm{Spec}\,(W[\![\mathbf{\Gamma}_\mathcal{K}]\!])(\overline{\mathbf{Q}}_p)$, the composition $\upsilon_P = P \circ \upsilon$ is the $p$-adic avatar $\widehat{\varphi}_P$ of an arithmetic Hecke character $\varphi_P$ of infinity type $(k(P)-1)\Sigma$. Enlarging $W$ if necessary, we suppose that $\widehat{\lambda}$ takes values in $W^\times$ and consider the $\Lambda$-adic Hecke character $\Psi$ given by the product $\widehat{\lambda}\upsilon : \mathrm{Cl}_\mathcal{K}(\mathfrak{C}^{(p)}\mathfrak{S}^\infty) \to W[\![\mathbf{\Gamma}_\mathcal{K}]\!]^\times$. By a change of variable of $\Lambda_W$, we can suppose that there exists an arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ of weight $k(P) = k+1$ such that the arithmetic specialisation $\Psi_P$ equals $\widehat{\lambda}$. We summarise the consideration as follows.

**Proposition 4.4** *Suppose that $\lambda$ is an arithmetic Hecke character over $\mathcal{K}$ with infinity type $k\Sigma$ for $k > 1$ and $p$-adic avatar $\widehat{\lambda}$. Then, there exists an $\mathbb{I}$-adic Hecke character $\Psi$ and an arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ of weight $k(P) = k+1$ such that the arithmetic specialisation $\Psi_P$ equals $\widehat{\lambda}$.*

### 4.3 Deformation of modular measures

In this subsection, we construct a modular measure associated with a class of $\Lambda$-adic Hilbert modular forms. This gives a $\Lambda$-adic interpolation of the modular measures associated with a class of classical Hilbert modular forms in Sect. 4.1. In particular, this gives $p$-adic deformation of the modular measures. We also discuss the non-triviality of the measure.

Let the notation and hypotheses be as in Sect. 1–3. Let $\Psi$ be a $\mathbb{I}$-adic Hecke character as in Sect. 4.2. Let $F \in M_\Lambda(K_0(\mathfrak{l}), \mathbb{I})$, for a finite rank domain $\mathbb{I}$ over $\Lambda_W$ such that the following conditions are satisfied:

(MI1) $F$ is an $U_\mathfrak{l}$ eigenform with the eigenvalue $a_\mathfrak{l}(F) \in \mathbb{I}^\times$ and
(MI2) $F(x_n(ta)) = \Psi(a)^{-1}F(x_n(t))$ for $a \in U_n\mathcal{F}^\times$.

Specialising at an arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, we have:

(MP1) $f_P$ is an $U_\mathfrak{l}$ eigenform with the eigenvalue $(a_\mathfrak{l}(F))_P \in \overline{\mathbf{Z}}_p^\times$ and
(MP2) $f_P(x_n(ta)) = \Psi_P(a)^{-1}f_P(x_n(t))$ for $a \in U_n\mathcal{F}^\times$.

Let $\mathrm{d}\varphi_F$ be the measure on $\Gamma_\mathfrak{l}^-$ such that for $\phi : \Gamma_n^- \to \overline{\mathbb{I}}$, we have

$$\int_{\Gamma_\mathfrak{l}^-} \phi \mathrm{d}\varphi_F = (a_\mathfrak{l}(F))^{-n} \sum_{[t]_n \in \Gamma_n^-} F(x_n(t))\Psi(t)\phi([t]_n). \tag{4.4}$$

By a similar argument as in Sect. 4.1, the measure $\mathrm{d}\varphi_F$ is well defined in view of the conditions (MI1) and (MI2).

For $\phi : \Gamma_n^- \to \overline{\mathbf{Z}}_p$, replacing $\mathbb{I}$ by a finite flat extension we suppose that the right-hand side of (4.4) belongs to $\mathbb{I}$. Specialising at an arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, we have:

$$\left(\int_{\Gamma_\mathfrak{l}^-} \phi \mathrm{d}\varphi_F\right)_P = ((a_\mathfrak{l}(F))_P)^{-n} \sum_{[t]_n \in \Gamma_n^-} f_P(x_n(t))\Psi_P(t)\phi([t]_n). \tag{4.5}$$

From the definition (4.1) and (4.5), we conclude that

$$\left(\int_{\Gamma_\mathfrak{l}^-} \phi \mathrm{d}\varphi_F\right)_P = \int_{\Gamma_\mathfrak{l}^-} \phi \mathrm{d}\varphi_{f_P}. \tag{4.6}$$

Here the right-hand side is defined using the $p$-adic Hecke character $\Psi_P$ (cf. (4.1)). In the sense of (4.6), the $\Lambda$-adic measure $\mathrm{d}\varphi_F$ interpolates the classical measures $\mathrm{d}\varphi_{f_P}$.

From now, let $\nu \in \mathfrak{X}_\mathfrak{l}^-$ and $\int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_F \in \mathbb{I}$. The following is an immediate consequence of Weierstrass preparation theorem and Theorem 4.1.

**Proposition 4.5** *Suppose that there exists an arithmetic* $P_0 \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ *such that the hypothesis (H) holds for the classical Hilbert modular form* $f_{P_0}$*. Then, for "almost all"* $\nu \in \mathfrak{X}_\mathfrak{l}{}^\times$*, we have* $\int_{\Gamma_\mathfrak{l}^-} \nu d\varphi_F \in \mathbb{I}^\times$*. In particular,* $\int_{\Gamma_\mathfrak{l}^-} \nu d\varphi_{f_P} \in \overline{\mathbf{Z}}_p^\times$ *for all arithmetic* $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$.

This can be considered as an $\mathbb{I}$-adic analogue of Theorem 4.1.

*Remark* We would like to state the following question.

(Q) Let $\nu \in \mathfrak{X}_\mathfrak{l}^-$ be given. When is $\int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_F \neq 0$?

For "almost all" $\nu$, the above proposition gives a criteria. However, for a given $\nu$ this seems to be a subtle question and perhaps requires new ideas.

## 5 Interpolation of genuine $p$-adic L-functions

In this section, we prove the existence of an $l \neq p$-interpolation of the genuine $p$-adic L-function of the $\mathbb{I}$-adic Hecke character over the $\mathfrak{l}$-power order anticyclotomic characters (cf. Theorem A). We also describe the non-triviality.

This section builds on the earlier section about deformation of modular measures. The construction of $\mathbb{I}$-adic family toric Eisenstein series constitutes a significant part of the section.

### 5.1 Toric Eisenstein series

In Sects. 5.1.1–5.1.4, we construct a classical toric Eisenstein series which will be used to construct anticyclotomic measure in Sect. 5.3.

We follow Hsieh's construction in [17] and [16] and make an appropriate choice for the local sections at the places dividing $p$. The latter allows to treat the case of conductor being divisible by places above $p$. In consideration of $\Lambda$-adic family of Hecke characters, the conductor of the corresponding arithmetic Hecke characters is indeed divisible by places above $p$. We closely follow the exposition in [17] and [16].

Let the notation and hypotheses be as in the introduction and Sect. 3.

#### *5.1.1 Eisenstein series on* $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$

In this subsection, we briefly recall the construction of an Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$ in terms of a section.

Let $\chi$ be an arithmetic Hecke character of infinity type $k\Sigma$, where $k \geq 1$.

We will identify the CM type $\Sigma \subset \mathrm{Hom}(\mathcal{K}, \mathbf{C})$ with the set $\mathrm{Hom}(\mathcal{F}, \mathbf{R})$ of Archimedean places of $\mathcal{F}$ by the restriction map. Let $K_\infty^0 := \prod_{\sigma\in\Sigma} \mathrm{SO}(2, \mathbf{R})$ be a maximal compact subgroup of $\mathrm{GL}_2(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$. We put

$$\chi^* = \chi|\cdot|_{\mathbf{A}_{\mathcal{K}}}^{-\frac{1}{2}} \quad \text{and} \quad \chi_+ = \chi|_{\mathbf{A}_{\mathcal{F}}^\times}.$$

For $s \in \mathbf{C}$, we let $I(s, \chi_+)$ denote the space consisting of smooth and $K_\infty^0$-finite functions $\phi : \mathrm{GL}_2(\mathbf{A}_{\mathcal{F}}) \to \mathbf{C}$ such that

$$\phi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g\right) = \chi_+^{-1}(d)\left|\frac{a}{d}\right|_{\mathbf{A}_{\mathcal{F}}}^s \phi(g).$$

Conventionally, the functions in $I(s, \chi_+)$ are called *sections*. Let $B$ be the upper triangular subgroup of $\mathrm{GL}_2$ and $N$ the upper unipotent subgroup. The adelic Eisenstein series associated with a section $\phi \in I(s, \chi_+)$ is defined by

$$E_{\mathbf{A}}(g, \phi) = \sum_{\gamma\in B(\mathcal{F})\backslash \mathrm{GL}_2(\mathcal{F})} \phi(\gamma g).$$

It is known that the series $E_{\mathbf{A}}(g, \phi)$ is absolutely convergent for $\Re s \gg 0$.

#### *5.1.2 Fourier coefficients of Eisenstein series*

In this subsection, we recall the formula for the Fourier coefficients of the Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$.

Put $\mathbf{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $v$ be a place of $\mathcal{F}$ and let $I_v(s, \chi_+)$ be the local constitute of $I(s, \chi_+)$ at $v$. For $\phi_v \in I_v(s, \chi_+)$ and $\beta \in \mathcal{F}_v$, we recall that the $\beta$th local Whittaker integral $W_\beta(\phi_v, g_v)$ is defined by

$$W_\beta(\phi_v, g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w}\begin{bmatrix}1 & x_v\\ 0 & 1\end{bmatrix} g_v)\psi(-\beta x_v)\mathrm{d}x_v,$$

and the intertwining operator $M_{\mathbf{w}}$ is defined by

$$M_{\mathbf{w}}\phi_v(g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w}\begin{bmatrix}1 & x_v\\ 0 & 1\end{bmatrix} g_v)\mathrm{d}x_v.$$

Here $\mathrm{d}x_v$ is Lebesgue measure if $\mathcal{F}_v = \mathbf{R}$ and is the Haar measure on $\mathcal{F}_v$ normalised so that $\mathrm{vol}(\mathcal{O}_{\mathcal{F}_v}, \mathrm{d}x_v) = 1$ if $\mathcal{F}_v$ is non-Archimedean. By definition, $M_{\mathbf{w}}\phi_v(g_v)$ is the 0th local Whittaker integral. It is well known that local Whittaker integrals converge absolutely for $\Re s \gg 0$ and have meromorphic continuation to all $s \in \mathbf{C}$.

If $\phi = \otimes_v\phi_v$ is a decomposable section, then it is well known that $E_{\mathbf{A}}(g, \phi)$ has the following Fourier expansion:

$$\begin{aligned} E_{\mathbf{A}}(g, \phi) &= \phi(g) + M_{\mathbf{w}}\phi(g) + \sum_{\beta\in\mathcal{F}} W_\beta(E_{\mathbf{A}}, g), \text{ where} \\ M_{\mathbf{w}}\phi(g) &= \frac{1}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}} \cdot \prod_v M_{\mathbf{w}}\phi_v(g_v); \quad W_\beta(E_{\mathbf{A}}, g) = \frac{1}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}} \cdot \prod_v W_\beta(\phi_v, g_v). \end{aligned} \tag{5.1}$$

#### *5.1.3 Choice of the local sections*

In this subsection, we choose the local sections which gives rise to the toric Eisenstein series (cf. [17, § 4.3]). As we consider the case of conductor possibly divisible by places above $p$, we choose the section at these places carefully.

We begin with some notation. Let $v$ be a place of $\mathcal{F}$. Let $F = \mathcal{F}_v$ (resp. $E = \mathcal{K} \otimes_{\mathcal{F}} \mathcal{F}_v$). Denote by $z \mapsto \bar{z}$ the complex conjugation. Let $|\cdot|$ be the standard absolute values on $F$ and let $|\cdot|_E$ be the absolute value on $E$ given by $|z|_E := |z\bar{z}|$. Let $d_F = d_{\mathcal{F}_v}$ be a fixed generator of the different $\mathfrak{d}_{\mathcal{F}}$ of $\mathcal{F}/\mathbf{Q}$. Write $\chi$ (resp. $\chi_+$) for $\chi_v$ (resp. $\chi_{+,v}$). If $v \in \mathbf{h}$, denote by $\varpi_v$ a uniformiser of $\mathcal{F}_v$. For a set $Y$, denote by $\mathbb{I}_Y$ the characteristic function of $Y$.

Suppose that $\mathfrak{C}$ is the prime-to-$p$ conductor of $\chi$ and $\mathfrak{l} \nmid \mathfrak{C}$. We write $\mathfrak{C} = \mathfrak{C}^+\mathfrak{C}^-$ such that $\mathfrak{C}^+$ (resp. $\mathfrak{C}^-$) is a product of prime factors split (resp. non-split) over $\mathcal{F}$. We further decompose $\mathfrak{C}^+ = \mathfrak{F}\mathfrak{F}_c$ such that $(\mathfrak{F}, \mathfrak{F}_c) = 1$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. Let $D_{\mathcal{K}/\mathcal{F}}$ be the discriminant of $\mathcal{K}/\mathcal{F}$ and let

$$\mathfrak{D} = p\mathfrak{l}\mathfrak{C}\mathfrak{C}^c D_{\mathcal{K}/\mathcal{F}}.$$

*Case I*: $v \nmid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$. We first suppose that $v = \sigma \in \Sigma$ is Archimedean and $F = \mathbf{R}$. For $g = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \mathrm{GL}_2(\mathbf{R})$, we put $J(g, i) := ci + d$. Define the sections $\phi^h_{k,s,\sigma}$ of weight $k$ in $I_v(s, \chi_+)$ by

$$\phi^h_{k,s,\sigma}(g) = J(g, i)^{-k}\left|\det(g)\right|^s \cdot \left|J(g, i)\overline{J(g, i)}\right|^{-s}.$$

Suppose that $v$ is non-Archimedean. Denote by $\mathcal{S}(F)$ and (resp. $\mathcal{S}(F \oplus F)$) the space of Bruhat–Schwartz functions on $F$ (resp. $F \oplus F$). Recall that the Fourier transform $\widehat{\varphi}$ for $\varphi \in \mathcal{S}(F)$ is defined by

$$\widehat{\varphi}(y) = \int_F \varphi(x)\psi(yx)\mathrm{d}x.$$

For a character $\mu : F^\times \to \mathbf{C}^\times$, we define a function $\varphi_\mu \in \mathcal{S}(F)$ by

$$\varphi_\mu(x) = \mathbb{I}_{O_v^\times}(x)\mu(x).$$

If $v = \mathfrak{l}$, let $\phi_{\chi,v,s}$ be the unique $N(O_v)$-invariant section supported in $B(F)\mathbf{w}N(O_v^\times)$ such that $\phi_{\chi,v,s}(\mathbf{w}) = 1$. The section $\phi_{\chi,s,v}$ is an $U_\mathfrak{l}$-eigenform with the eigenvalue $\chi_+(\varpi_\mathfrak{l})^{-1}$ (cf. [16, §4.3.5]).

If $v|p\mathfrak{F}\mathfrak{F}^c$ is split in $\mathcal{K}$, write $v = w\overline{w}$ with $w|\mathfrak{F}\Sigma_p$, and set

$$\varphi_w = \varphi_{\chi_w} \quad \text{and} \quad \varphi_{\overline{w}} = \varphi_{\chi_{\overline{w}}^{-1}}.$$

To a Bruhat–Schwartz function $\Phi \in \mathcal{S}(F \oplus F)$, we can associate a Godement section $f_{\Phi,s} \in I_v(s, \chi_+)$ defined by

$$f_{\Phi,s}(g) := \left|\det g\right|^s \int_{F^\times} \Phi((0,x)g)\chi_+(x)\,|x|^{2s}\,\mathrm{d}^\times x, \tag{5.2}$$

where $\mathrm{d}^\times x$ is the Haar measure on $F^\times$ such that $\mathrm{vol}(\mathcal{O}_F^\times, \mathrm{d}^\times x) = 1$. Define Godement sections by

$$\phi_{\chi,s,v} = f_{\Phi_v^0,s}, \quad \text{where } \Phi_v^0(x,y) = \begin{cases} \mathbb{I}_{O_v}(x)\mathbb{I}_{d_F^{-1}O_v}(y) & \cdots v \nmid \mathfrak{D}, \\ \varphi_{\overline{w}}(x)\widehat{\varphi}_w(y) & \cdots v|\mathfrak{F}\mathfrak{F}^c. \end{cases} \tag{5.3}$$

Let $u \in \mathcal{O}_F^\times$. Let $\varphi_{\overline{w}}^1$ and $\varphi_w^{[u]} \in \mathcal{S}(F)$ be the Bruhat–Schwartz functions defined by

$$\varphi_{\overline{w}}^1(x) = \mathbb{I}_{1+\varpi_v O_v}(x)\chi_{\overline{w}}^{-1}(x) \quad \text{and} \quad \varphi_w^{[u]}(x) = \mathbb{I}_{u(1+\varpi_v O_v)}(x)\chi_w(x).$$

Define $\Phi_v^{[u]} \in \mathcal{S}(F \oplus F)$ by

$$\Phi_v^{[u]}(x,y) = \frac{1}{\mathrm{vol}(1+\varpi_v O_v, \mathrm{d}^\times x)}\varphi_{\overline{w}}^1(x)\widehat{\varphi}_w^{[u]}(y) = (|\varpi_v|^{-1} - 1)\varphi_{\overline{w}}^1(x)\widehat{\varphi}_w^{[u]}(y). \tag{5.4}$$

*Case II*: $v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}^-$. In this case, $E$ is a field. We define an embedding $E \hookrightarrow M_2(F)$ by

$$a + b\delta \mapsto \begin{bmatrix} a & b\delta^2 \\ b & a \end{bmatrix}.$$

Here $\delta$ is as in [12, $(d1)$ and $(d2)$]. Then $\mathrm{GL}_2(F) = B(F)\rho(E^\times)$. We fix a $O_v$-basis $\{1, \boldsymbol{\theta}_v\}$ of $\mathcal{O}_E$ such that $\boldsymbol{\theta}_v$ is a uniformiser if $v$ is ramified and $\overline{\boldsymbol{\theta}_v} = -\boldsymbol{\theta}_v$ if $v \nmid 2$. Let $t_v = \boldsymbol{\theta}_v + \overline{\boldsymbol{\theta}_v}$ and put

$$\varsigma_v = \begin{bmatrix} d_{\mathcal{F}_v} & -2^{-1}t_v \\ 0 & d_{\mathcal{F}_v}^{-1} \end{bmatrix}.$$

Let $\phi_{\chi,s,v}$ be the smooth section in $I_v(s, \chi_+)$ defined by

$$\phi_{\chi,s,v}\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\rho(z)\varsigma_v\right) = L(s,\chi_v)\cdot\chi_+^{-1}(d)\left|\frac{a}{d}\right|^s \cdot \chi^{-1}(z) \quad (b \in B(F),\ z \in E^\times). \tag{5.5}$$

Here $L(s, \chi_v)$ is the local Euler factor of $\chi_v$.

#### *5.1.4 q-Expansion of normalised Eisenstein series*

In this subsection, we describe the formula for the $q$-expansion coefficients of the classical toric Eisenstein series arising from normalisation of the local sections in Sect. 5.1.3.

Let $\mathcal{U}_p$ be the torsion subgroup of $\mathcal{O}_{\mathcal{F}_p}^\times$. For $u = (u_v)_{v|p} \in \mathcal{U}_p$, let $\Phi_p^{[u]} = \otimes_{v|p}\Phi_v^{[u_v]}$ be the Bruhat–Schwartz function defined in (5.4). Define the section $\phi_{\chi,s}^h(\Phi_p^{[u]}) \in I(s, \chi_+)$ by

$$\phi_{\chi,s}^h(\Phi_p^{[u]}) = \bigotimes_{\sigma\in\Sigma}\phi_{k,s,\sigma}^h \bigotimes_{\substack{v\in\mathbf{h},\\ v\nmid p}}\phi_{\chi,s,v} \bigotimes_{v|p} f_{\Phi_v^{[u_v]},s}.$$

We put

$$X^+ = \left\{\tau = (\tau_\sigma)_{\sigma\in\Sigma} \in \mathbf{C}^\Sigma \mid \mathrm{Im}\tau_\sigma > 0 \text{ for all } \sigma \in \Sigma\right\}.$$

The holomorphic Eisenstein series $\mathbb{E}^h_{\chi,u} : X^+ \times \mathrm{GL}_2(\mathbf{A}_{\mathcal{F},f}) \to \mathbf{C}$ is defined by

$$\mathbb{E}^h_{\chi,u}(\tau, g_f) := \frac{\Gamma_\Sigma(k\Sigma)}{\sqrt{|D_\mathcal{F}|_\mathbf{R}}(2\pi i)^{k\Sigma}} \cdot E_\mathbf{A}\left((g_\infty, g_f), \phi^h_{\chi,s}(\Phi_p^{[u]})\right)|_{s=0} \cdot \prod_{\sigma\in\Sigma} J(g_\sigma, i)^k, \qquad (5.6)$$
$$(g_\infty = (g_\sigma)_\sigma \in \mathrm{GL}_2(\mathcal{F}\otimes_\mathbf{Q}\mathbf{R}),\ (g_\sigma i)_{\sigma\in\Sigma} = (\tau_\sigma)_{\sigma\in\Sigma}).$$

We choose $N = \mathbf{N}_{\mathcal{K}/\mathbf{Q}}(\mathfrak{C})^m$ for a sufficiently large integer $m$ so that $\phi_{\chi,s,v}$ are invariant by $U(N)$ for every $v|\mathfrak{C}\mathfrak{C}^c$, and put $K := U(N)$. Then, the section $\phi_{\chi,s}(\Phi_p^{[u]})$ is invariant by $K_1^n$ for a sufficiently large $n$. Then, the holomorphic Eisenstein series $\mathbb{E}^h_{\chi,u}$ has parallel weight $k$ and level $K_1^n$.

**Proposition 5.1** *(Hsieh) Let $\mathbf{c} = (\mathbf{c}_v) \in (\mathbf{A}_{\mathcal{F},f})^\times$ such that $\mathbf{c}_v = 1$ at $v|\mathfrak{D}$ and let $\mathfrak{c} = \mathbf{c}(\mathcal{O}_\mathcal{F}\otimes_\mathbf{Z}\widehat{\mathbf{Z}}) \cap \mathcal{F}$. Say, the q-expansion of $\mathbb{E}^h_{\chi,u}$ at the cusp $(O, \mathfrak{c}^{-1})$ is given by*

$$\mathbb{E}^h_{\chi,u}|_{(O,\mathfrak{c}^{-1})}(q) = \sum_{\beta\in\mathcal{F}_+} \mathbf{a}_\beta(\mathbb{E}^h_{\chi,u}, \mathfrak{c}) \cdot q^\beta.$$

*Then, the $\beta$th Fourier coefficient $\mathbf{a}_\beta(\mathbb{E}^h_{\chi,u}, \mathfrak{c})$ is given by*

$$\mathbf{a}_\beta(\mathbb{E}^h_{\chi,u}, \mathfrak{c}) = \beta^{(k-1)\Sigma}(|\mathbf{c}|_\mathfrak{l}\mathbb{I}_{O_\mathfrak{l}}(\beta\mathbf{c}))\prod_{w|\mathfrak{F}}\chi_w(\beta)\mathbb{I}_{O_v^\times}(\beta)\prod_{w\in\Sigma_p}\chi_w(\beta)\mathbb{I}_{u_v(1+\varpi_v O_v)}(\beta)$$
$$\times\prod_{v\nmid\mathfrak{D}}\left(\sum_{i=0}^{v(\mathbf{c}_v\beta)}\chi^*(\varpi_v^i)\right)\cdot\prod_{v|\mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}} L(0,\chi_v)\widetilde{A}_\beta(\chi_v),$$

*where*

$$\widetilde{A}_\beta(\chi_v) = \int_{\mathcal{F}_v}\chi_v^{-1}|\cdot|_E^s(x_v+\boldsymbol{\theta}_v)\psi(-d_{\mathcal{F}_v}^{-1}\beta x_v)dx_v|_{s=0}$$
$$:= \lim_{n\to\infty}\int_{\varpi_v^{-n}\mathcal{O}_{\mathcal{F}_v}}\chi_v^{-1}(x_v+\boldsymbol{\theta}_v)\psi(-d_{\mathcal{F}_v}^{-1}\beta x_v)dx_v. \qquad (5.7)$$

*Moreover, suppose that either of the following holds:*

*(1).* $k > 2$,
*(2).* $\mathfrak{f} \neq O$,
*(3).* $\chi_+ = \tau_{\mathcal{K}/\mathcal{F}}|\cdot|_{\mathbf{A}_\mathcal{F}}$.

*Then, $\mathbb{E}^h_{\chi,u} \in M_k(K_0(\mathfrak{l}), \overline{\mathbf{Z}}_{(p)})$.*

*Proof* This follows from (3.1) and the calculations of local Whittaker integrals of special local sections in [16, § 4.3] (cf. [17, Prop. 4.1 and Prop. 4.4] and [16, Prop. 4.7]). □

Let $\mathbb{E}^h_\chi = \sum_{u\in\mathcal{U}_p}\mathbb{E}^h_{\chi,u}$ and $\mathbf{D}_\mathfrak{C} = \prod_{v|\mathfrak{C}^- \mathcal{D}_{\mathcal{K}/\mathcal{F}}}\mathcal{K}_v^\times$. The Eisenstein series $\mathbb{E}^h_\chi$ satisfies the following transformation properties.

**Lemma 5.2** *(1). $\mathbb{E}^h_\chi \in M_k(K_0(\mathfrak{l}), \overline{\mathbf{Z}}_{(p)})$ is an $U_\mathfrak{l}$-eigenform with the eigenvalues $\chi_+(\varpi_\mathfrak{l})$.*
*(2). $\mathbb{E}^h_\chi|[r] = \widehat{\chi}^{-1}(r)\mathbb{E}^h_\chi$, where $r \in \mathbf{D}_\mathfrak{C}$.*
*(3). $\mathbb{E}^h_\chi(x_n(ta)) = \widehat{\chi}^{-1}(a)\mathbb{E}^h_\chi(x_n(t))$, where $a \in \mathbf{D}_\mathfrak{C}\mathbf{A}^\times_\mathcal{F}U_n$ (cf. [16, Prop. 4.8]).*

### 5.2 $\Lambda$-Adic toric Eisenstein series

In this subsection, we describe an $\mathbb{I}$-adic toric Eisenstein series which interpolates the classical toric Eisenstein series in the previous subsection.

Let the notation and hypotheses be as in Sects. 4.2 and 5.1. In particular, $\Psi$ is an $\mathbb{I}$-adic Hecke character with conductor prime-to-$\mathfrak{l}$ with the prime to-$p$ conductor $\mathfrak{C}^{(p)}(\Psi)$ $\mathfrak{l} \nmid \mathfrak{C}^{(p)}(\Psi)$. For simplicity, we denote it by $\mathfrak{C}(\Psi)$. We decompose $\mathfrak{C}(\Psi) = \mathfrak{C}(\Psi)^+\mathfrak{C}(\Psi)^-$ and define $\mathfrak{D}(\Psi)$ accordingly as in Sect. 5.1.3. Let $\mathfrak{R}(\Psi)'$ be the product of ramified places $v$ where $\Psi_v$ is unramified. For $v|\mathfrak{C}(\Psi)^-\mathcal{D}_{\mathcal{K}/\mathcal{F}}$, let

$$L(0, \Psi_v) = \begin{cases} 1 & \text{if } v|\mathfrak{C}(\Psi)^- \\ \frac{1}{1-\Psi_v(\varpi_v)} & \text{if } v|\mathfrak{R}(\Psi)'. \end{cases} \tag{5.8}$$

Recall that for an arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, the specialisation $\Psi_P$ is the $p$-adic avatar of an arithmetic Hecke character. For $u \in \mathcal{U}_p$, we have the corresponding toric Eisenstein series $\mathbb{E}^h_{\Psi_P,u}$ (cf. Sect. 5.1). As $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ varies over arithmetic primes, these classical Eisenstein series are interpolated by an $\mathbb{I}$-adic Eisenstein series as follows.

**Proposition 5.3** *Replacing $\mathbb{I}$ by a finite flat extension if necessary, there exists an $\mathbb{I}$-adic Eisenstein series $\mathcal{E}_{\Psi,u} \in M_\Lambda(K_0(\mathfrak{l}), \mathbb{I})$ such that for an arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, we have*

$$(\mathcal{E}_{\Psi,u})_P = \mathbb{E}^h_{\Psi_P,u}.$$

*Let $\mathbf{c} = (\mathbf{c}_v) \in (\mathbf{A}_{\mathcal{F},f})^\times$ such that $\mathbf{c}_v = 1$ at $v|\mathfrak{D}$ and let $\mathfrak{c} = \mathbf{c}(\mathcal{O}_\mathcal{F} \otimes_\mathbf{Z} \widehat{\mathbf{Z}}) \cap \mathcal{F}$. Say, the $q$-expansion of $\mathcal{E}_{\Psi,u}$ at the cusp $(O, \mathfrak{c}^{-1})$ is given by*

$$\mathcal{E}_{\Psi,u}|_{(O,\mathfrak{c}^{-1})}(q) = \sum_{\beta\in\mathcal{F}_+} \mathbf{a}_\beta(\mathcal{E}_{\Psi,u}, \mathfrak{c}) \cdot q^\beta.$$

*Then, the $\beta$th Fourier coefficient $\mathbf{a}_\beta(\mathcal{E}_{\Psi,u}, \mathfrak{c})$ is given by*

$$\mathbf{a}_\beta(\mathcal{E}_{\Psi,u}, \mathfrak{c}) = N_{\mathcal{F}/\mathbf{Q}}(\beta)^{-1}(|\mathbf{c}|_\mathfrak{l}\mathbb{I}_{O_\mathfrak{l}}(\beta\mathbf{c})) \prod_{w|\mathfrak{F}(\Psi)} \Psi_w(\beta)\mathbb{I}_{O_v^\times}(\beta) \prod_{w\in\Sigma_p} \Psi_w(\beta)\mathbb{I}_{u_v(1+\varpi_v O_v)}(\beta)$$
$$\times \prod_{v\nmid\mathfrak{D}(\Psi)} \left(\sum_{i=0}^{v(\mathbf{c}_v\beta)} \Psi^*(\varpi_v^i)\right) \cdot \prod_{v|\mathfrak{C}(\Psi)^-D_{\mathcal{K}/\mathcal{F}}} L(0, \Psi_v)\widetilde{A}_\beta(\Psi_v),$$

*where*

$$\begin{aligned}\widetilde{A}_\beta(\Psi_v) &= \int_{\mathcal{F}_v} \Psi_v^{-1}|\cdot|_E^s(x_v + \boldsymbol{\theta}_v)\psi\left(-d_{\mathcal{F}_v}^{-1}\beta x_v\right) \mathrm{d}x_v|_{s=0} \\ &:= \lim_{n\to\infty} \int_{\varpi_v^{-n}\mathcal{O}_{\mathcal{F}_v}} \Psi_v^{-1}(x_v + \boldsymbol{\theta}_v)\psi\left(-d_{\mathcal{F}_v}^{-1}\beta x_v\right) \mathrm{d}x_v.\end{aligned} \tag{5.9}$$

*Proof* We first note that the integral in (5.9) is well defined as it is indeed a finite sum (cf. [16, (4.17)]).

We need to show that $(\mathbf{a}_\beta(\mathcal{E}_{\Psi,u}, \mathfrak{c}))_P = \mathbf{a}_\beta(\mathcal{E}_{\Psi_P,u}, \mathfrak{c})$. In view of the fact that both of the coefficients are a product of local Whittaker integrals and Lemma 4.3, it suffices to show the equality for the local Whittaker integrals. For $v|p$, the equality from the fact that $\widehat{\lambda}_{\Sigma_p}(\beta) = \beta^{k\Sigma}\lambda_{\Sigma_p}(\beta)$, where $\lambda_\Sigma(\beta) = \prod_{w\in\Sigma_p}\lambda_w(\beta)$. For $v|\mathfrak{C}^-\mathcal{D}_{\mathcal{K}/\mathcal{F}}$, the equality follows form the formula for the local Euler factor $L(0, \Psi_{P,v})$ (cf. [16, 4.3.6]) and (5.8). For the other $v$, the equality follows immediately from the definitions. ◻

Let $\mathcal{E}_\Psi = \sum_{u \in \mathcal{U}_p} \mathcal{E}_{\Psi,u}$. The $\mathbb{I}$-adic toric Eisenstein series $\mathcal{E}_\Psi$ satisfies the following transformation properties.

**Lemma 5.4** *(1).* $\mathcal{E}_\Psi \in M_\Lambda(K_0(\mathfrak{l}), \mathbb{I})$ *is an* $U_\mathfrak{l}$*-eigenform with the eigenvalues* $\Psi_+(\varpi_\mathfrak{l})$.
*(2).* $\mathcal{E}_\Psi|[r] = \Psi^{-1}(r)\mathcal{E}_\Psi$, *where* $r \in \mathbf{D}_\mathfrak{C}$.
*(3).* $\mathcal{E}_\Psi(x_n(ta)) = \Psi^{-1}(a)\mathcal{E}_\Psi(x_n(t))$, *where* $a \in \mathbf{D}_\mathfrak{C}\mathbf{A}_\mathcal{F}^\times U_n$.

*Proof* From part (1) (resp. (2)) of Lemma 5.2, part (1) (resp. (2)) follows immediately. Part (3) follows from (Gp$\Lambda$) and part (3) of Lemma 5.2. □

### 5.3 Interpolation

In this subsection, we prove the existence of an $l \neq p$-interpolation of the genuine $p$-adic L-function of the $\mathbb{I}$-adic Hecke character over the $\mathfrak{l}$-power order anticyclotomic characters (cf. Theorem A). We also describe the non-triviality.

Let the notation and hypotheses be as in Sects. 1–4 and 5.1–5.2. We first state the following result due to Hida and Hsieh.

**Theorem 5.5** (Hida, Hsieh) *Let* $\lambda$ *be an arithmetic Hecke character of infinity type* $k\Sigma$, *with* $k \geq 1$ *and the conductor prime-to-*$\mathfrak{l}$. *There exists an* $\overline{\mathbf{Z}}_p$*-valued measure* $d\varphi_\lambda$ *on* $\Gamma_\mathfrak{l}$ *such that for* $\nu \in \mathfrak{X}_\mathfrak{l}^-$, *we have*

$$\frac{1}{\Omega_p^{k\Sigma}} \cdot \int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_\lambda \doteq L^{\mathrm{alg}, \mathfrak{l}p}(0, \lambda\nu) \tag{5.10}$$

*Here* $\doteq$ *denotes that the equality holds up to an explicit p-adic unit.*

*Proof* In view of (MC1)–(MC2) and Lemma 5.2, we have a well-defined measure $\mathrm{d}\varphi_{\mathbb{E}_\lambda^h}$ on $\Gamma_\mathfrak{l}^-$. We take $\mathrm{d}\varphi_\lambda$ to be $\mathrm{d}\varphi_{\mathbb{E}_\lambda^h}$. For $\nu \in \mathfrak{X}_\mathfrak{l}$, we have the following formula for the toric period integral (cf. [16, §5] and [17, §4.7 and §4.8]):

$$\frac{1}{\Omega_p^{k\Sigma}} \cdot \int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_{\mathbb{E}_\lambda^h} = L^{\mathrm{alg}, \mathfrak{l}p}(0, \lambda\nu) \cdot \frac{2^r L_\mathfrak{G} C_\mathfrak{F}}{\sqrt{|D_\mathcal{F}|_\mathbf{R}}}, \tag{5.11}$$

where $r$ is the number of prime factors of $D_{\mathcal{K}/\mathcal{F}}$ and $L_\mathfrak{G}, C_\mathfrak{F} \in \overline{\mathbf{Z}}_p^\times$ as in [16, §5]. □

We are now ready to prove Theorem A.

**Theorem A** *Let* $\mathcal{F}$ *be a totally real field and p an odd prime unramified in* $\mathcal{F}$. *Let* $l \neq p$ *be an odd prime unramified in* $\mathcal{F}$ *and* $\mathfrak{l}$ *a prime above l. Let* $\mathcal{K}/\mathcal{F}$ *be a CM quadratic extension with a p-ordinary CM type* $\Sigma$. *Let* $\Gamma_\mathfrak{l}^-$ *be the* $\mathfrak{l}$*-anticyclotomic Galois group over* $\mathcal{K}$. *Let* $\mathbb{I}$ *be a domain of finite rank over p-adic one-variable Iwasawa algebra and* $\overline{\mathbb{I}}$ *the integral closure of* $\mathbb{I}$ *as above. Let* $\Psi$ *be an* $\mathbb{I}$*-adic Hecke character over* $\mathcal{K}$ *with prime-to-p conductor prime-to-*$\mathfrak{l}$. *Then, there exists an* $\overline{\mathbb{I}}$*-valued measure* $d\varphi_\Psi$ *on* $\Gamma_\mathfrak{l}^-$ *such that*

$$\frac{1}{\Omega_p^{(k(P)-1)\Sigma}} \cdot \left(\int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_\Psi\right)_P = \frac{1}{\Omega_p^{(k(P)-1)\Sigma}} \cdot \int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_{\Psi_P} \doteq L^{\mathrm{alg}, \mathfrak{l}p}(0, \Psi_P\nu). \tag{5.12}$$

Here $\nu$ is a finite order character of $\Gamma_\mathfrak{l}^-$, $P \in \mathrm{Spec}\,(\mathbb{I}_\nu)(\overline{\mathbf{Q}}_p)$ an arithmetic prime with weight $k(P)$, $\mathbb{I}_\nu$ the finite flat extension of $\mathbb{I}$ and CM periods $(\Omega_p, \Omega_\infty)$ as above. In particular, $\int_{\Gamma_\mathfrak{l}^-} \nu \mathrm{d}\varphi_\Psi \in \mathbb{I}_\nu$ equals the genuine $p$-adic L-function of the $\mathbb{I}_\nu$-adic Hecke character $\Psi\nu$.

*Proof* In view of (ML1)–(ML2) and Lemma 5.4, we have a well-defined measure $\mathrm{d}\varphi_{\mathcal{E}_\Psi}$ on $\Gamma_{\mathfrak{l}}^{-}$. By construction, it interpolates the measures $\mathrm{d}\varphi_{\Psi_P}$ for arithmetic $P \in \mathrm{Spec}\,(\mathbb{I}_\nu)(\overline{\mathbf{Q}}_p)$. □

*Remark* In view of Proposition 4.4, the above theorem thus provides an $\mathbb{I}$-adic interpolation of Hida's modular associated with a given Hecke character of infinity type $k\Sigma$ for $k \geq 1$. The $p$-adic deformation in "prime-to-$p$ Iwasawa theory" does not seem to be considered in the literature before.

We now consider the non-triviality of the measure $\mathrm{d}\varphi_\Psi$. For simplicity, we only consider the case when $\Psi$ is self-dual (cf. Sect. 4.2 the for the definition).

For each $\nu|\mathfrak{C}(\Psi)^{-}$ and arithmetic $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$, let $\mu_p(\Psi_{P,\nu})$ be the local invariant given by

$$\mu_p(\Psi_{P,\nu}) = \inf_{x \in \mathcal{K}_\nu^\times} \nu_p(\Psi_P(x) - 1). \tag{5.13}$$

The following is an immediate consequences of the non-triviality results in [7,10] and [16].

**Proposition 5.6** *Let $\Psi$ be a self-dual $\mathbb{I}$-adic Hecke character with the prime-to-p conductor prime-to-$\mathfrak{l}$ and the root number one. Suppose that there exists an $P_0 \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ such that $\mu_p(\Psi_{P_0,\nu}) = 0$, for all $\nu|\mathfrak{C}(\Psi)^{-}$. Then, for "almost all" $\nu \in \mathfrak{X}_{\mathfrak{l}}{}^\times$, we have $\int_{\Gamma_{\mathfrak{l}}^{-}} \nu \mathrm{d}\varphi_\Psi \in \mathbb{I}_\nu^\times$.*

*Proof* In view of Proposition 4.5, it suffices to verify the hypothesis (H) for the toric Eisenstein series $\mathbb{E}^h_{\Psi_{P_0}}$. As $(\mathbb{E}^h_{\Psi_{P_0}})^{\mathfrak{R}} = |\Delta_{\mathrm{alg}}|\mathbb{E}^h_{\Psi_{P_0}}$, the hypothesis follows from [16, Prop. 6.3] and [17, Thm. 6.5]. □

## 6 Interpolation of *p*-adic Abel–Jacobi image

In this section, we briefly describe the upcoming analogous interpolation of the $p$-adic Abel–Jacobi image of generalised Heegner cycles associated with a self-dual Rankin–Selberg convolution of an $\mathbb{I}$-adic Hida family of elliptic Hecke eigenforms and an $\mathbb{I}$-adic Hecke character with root number $-1$.

Under the generalised Heegner hypothesis, a self-dual Rankin–Selberg convolution of an $\mathbb{I}$-adic Hida family of elliptic Hecke eigenforms and an $\mathbb{I}$-adic Hecke character has root number $-1$. Accordingly, the central-critical $L$-values identically vanish in the family. The Bloch–Beilinson conjectures thus predict the existence of non-torsion cycles associated with the family which are homologically trivial. A candidate for the cycles are generalised Heegner cycles. The construction is due to Bertolini–Darmon–Prasanna and generalises the one of classical Heegner cycles (cf. [1, §2] and [2, §4.1]). The cycle lives in a middle-dimensional Chow group of a fibre product of a Kuga–Sato variety arising from a modular curve and a self-product of a CM elliptic curve. In the case of weight two, the cycle coincides with a Heegner point and the $p$-adic Abel–Jacobi image with the $p$-adic formal group logarithm. In an ongoing work, we construct an analogous interpolation of the $p$-adic Abel–Jacobi image of the cycles.

For simplicity, we mostly restrict here to the case of Heegner points.

Unless otherwise stated, let the notation and hypotheses be as in the introduction. Let $\mathcal{K}/\mathbf{Q}$ be an imaginary quadratic extension and $\mathcal{O}$ the ring of integers. Let $p$ be an odd prime split in $\mathcal{K}$. For a positive integer $n$, let $\mathcal{H}_n$ be the ring class field of $\mathcal{K}$ with conductor

$n$. Let $\mathcal{H}$ be the Hilbert class field. Let $N$ be a positive integer such that $p \nmid N$. For $k \geq 2$, let $S_k(\Gamma_0(N), \epsilon)$ be the space of elliptic modular forms of weight $k$, level $\Gamma_0(N)$ and Neben character $\epsilon$. Let $f \in S_2(\Gamma_0(N), \epsilon)$ be a Hecke eigenform and $\mathcal{E}_f$ the corresponding Hecke field. Let $N_\epsilon | N$ be the conductor of $\epsilon$.

We assume the following generalised Heegner hypothesis:

(Hg) $\mathcal{O}$ contains an ideal $\mathfrak{N}$ of norm $N$ such that $\mathcal{O}/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$.

From now, we fix such an ideal $\mathfrak{N}$. Let $\mathfrak{N}_\epsilon | \mathfrak{N}$ be the unique ideal of norm $N_\epsilon$.

Let $\mathbf{N}$ denote the norm Hecke character over $\mathbf{Q}$ and $\mathbf{N}_{\mathcal{K}} := \mathbf{N} \circ N_{\mathbf{Q}}^{\mathcal{K}}$ the norm Hecke character over $\mathcal{K}$. For a Hecke character $\lambda$ over $\mathcal{K}$, let $\mathfrak{f}_\lambda$ (resp. $\epsilon_\lambda$) denote its conductor (resp. the restriction $\lambda|_{\mathbf{A}_{\mathbf{Q}}^\times}$). We say that $\lambda$ is central-critical for $f$ if it is of infinity type $(j_1, j_2)$ with $j_1 + j_2 = 2$ and $\epsilon_\lambda = \epsilon \mathbf{N}^2$.

Let $b$ be a positive integer prime to $N$. Let $\Sigma_{cc}(b, \mathfrak{N}, \epsilon)$ be the set of Hecke characters $\lambda$ such that:

(C1) $\lambda$ is central-critical for $f$,
(C2) $\mathfrak{f}_\lambda = b \cdot \mathfrak{N}_\epsilon$ and
(C3) The local root number $\epsilon_q(f, \lambda^{-1}) = 1$, for all finite primes $q$.

Let $\chi$ be a finite order Hecke character such that $\chi \mathbf{N}_{\mathcal{K}} \in \Sigma_{cc}(b, \mathfrak{N}, \epsilon)$. Let $\mathcal{E}_{f,\chi}$ be the finite extension of $\mathcal{E}_f$ obtained by adjoining the values of $\chi$.

Let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$ and the cusp $\infty$ of $X_1(N)$. Let $J_1(N)$ be the corresponding Jacobian. Let $B_f$ be the abelian variety associated with $f$ by the Eichler–Shimura correspondence and $\Phi_f : J_1(N) \to B_f$ the associated surjective morphism. By possibly replacing $B_f$ with an isogenous abelian variety, we suppose that $B_f$ has endomorphisms by the integer ring $\mathcal{O}_{\mathcal{E}_f}$. Let $\omega_f$ be the differential form on $X_1(N)$ corresponding to $f$. We use the same notation for the corresponding 1-form on $J_1(N)$. Let $\omega_{B_f} \in \Omega^1(B_f/\mathcal{E}_f)^{\mathcal{O}_{\mathcal{E}_f}}$ be the unique 1-form such that $\Phi_f^*(\omega_{B_f}) = \omega_f$. Here $\Omega^1(B_f/\mathcal{E}_f)^{\mathcal{O}_{\mathcal{E}_f}}$ denotes the subspace of 1-forms stable under the action of the integer ring $\mathcal{O}_{\mathcal{E}_f}$.

Recall that $b$ is a positive integer prime to $N$. Let $A_b$ be an elliptic curve with endomorphism ring $\mathcal{O}_b = \mathbf{Z} + b\mathcal{O}$, defined over the ring class field $H_b$. Let $t$ be a generator of $A_b[\mathfrak{N}]$. We thus obtain a point $x_b = (A_b, A_b[\mathfrak{N}], t) \in X_1(N)(\mathcal{H}_{bN})$. Let $\Delta_b = [A_b, A_b[\mathfrak{N}], t] - (\infty) \in J_1(N)(\mathcal{H}_{bN})$ be the corresponding Heegner point on the modular Jacobian. We regard $\chi$ as a character $\chi : \mathrm{Gal}(\mathcal{H}_{bN}/\mathcal{K}) \to \mathcal{E}_{f,\chi}$. Let $\mathcal{H}_\chi$ be the abelian extension of $\mathcal{K}$ cut out by the character $\chi$. To the pair $(f, \chi)$, we associate the Heegner point $P_f(\chi)$ given by

$$P_f(\chi) = \sum_{\sigma \in \mathrm{Gal}(\mathcal{H}_{bN}/\mathcal{K})} \chi^{-1}(\sigma) \Phi_f(\Delta_b^\sigma) \in B_f(\mathcal{H}_\chi) \otimes_{\mathcal{O}_{\mathcal{E}_f}} \mathcal{E}_{f,\chi}. \tag{6.1}$$

A natural invariant associated with the Heegner points is the corresponding $p$-adic formal group logarithm. The restriction of the $p$-adic logarithm gives a homomorphism

$$\log_{\omega_{B_f}} : B_f(\mathcal{H}_\chi) \to \mathbf{C}_p.$$

We extend it to $B_f(\mathcal{H}_\chi) \otimes_{\mathcal{O}_{\mathcal{E}_f}} \mathcal{E}_{f,\chi}$ by $\mathcal{E}_{f,\chi}$-linearity.

Let $F$ be an $\mathbb{I}$-adic Hida family of elliptic Hecke eigenforms and $\Psi$ an $\mathbb{I}$-adic Hecke character over $\mathcal{K}$ of prime-to-$p$ conductor $c\mathfrak{N}_\epsilon$ for an integer $c$ such that the corresponding Rankin–Selberg convolution is self-dual. As $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ varies over arithmetic primes of weight 2, we consider the variation of the $p$-adic formal group logarithm of the corresponding Heegner points.

Our result regarding the variation is the following.

**Theorem B** *Let the notation and hypotheses be as above. In addition to (ord), suppose that the generalised Heegner hypothesis (Hg) holds. Let $F$ be an $\mathbb{I}$-adic Hida family of elliptic Hecke eigenforms and $\Psi$ an $\mathbb{I}$-adic Hecke character over $\mathcal{K}$ of prime-to-$p$ conductor $c\mathfrak{N}_\epsilon$ for an integer $c$ such that the corresponding Rankin–Selberg convolution is self-dual. Let $l \neq p$ be an odd prime unramified in $K$ and prime-to-$cN$. Then, there exists an $\overline{\mathbb{I}}$-valued measure $d\varphi_{F,\Psi}$ on $\Gamma_l^-$ such that*

$$\frac{1}{\Omega_p} \cdot \left( \int_{\Gamma_l^-} \nu \mathrm{d}\varphi_{F,\Psi} \right)_P \doteq \log_{\omega_{B_{f_P}}} P_{f_P}(\nu \mathbf{N}^{-1} \Psi_P), \tag{6.2}$$

*where $P \in \mathrm{Spec}\,(\mathbb{I}_\nu)(\overline{\mathbf{Q}}_p)$ is an arithmetic prime of weight 2 and $\doteq$ denotes that the equality holds up to an explicit $p$-adic factor.*

We also study the non-triviality of the measure $\mathrm{d}\varphi_{F,\Psi}$ based on the non-triviality results in [3].

We now briefly describe the strategy of the proof. In view of the $p$-adic Waldspurger formula in [1] and [2], the $p$-adic formal logarithms of the Heegner points associated with the weight two specialisation $f_P$ essentially equals a weighted sum of evaluation of a toric weight zero $p$-adic modular form $g_P$ at the $l$-power conductor CM points. As $P \in \mathrm{Spec}\,(\mathbb{I}_\nu)(\overline{\mathbf{Q}}_p)$ varies over arithmetic primes, we prove that these weight zero $p$-adic modular forms $g_P$ are interpolated by an $\mathbb{I}_\nu$-adic elliptic modular form $G_F$. The $\mathbb{I}$-adic measure $\mathrm{d}\varphi_{F,\Psi}$ is then constructed by the technique in Sect. 4.3 applied to the $\mathbb{I}_\nu$-adic elliptic modular form $G_F$ and the $\mathbb{I}_\nu$-adic Hecke character $\Psi$.

As $P \in \mathrm{Spec}\,(\mathbb{I})(\overline{\mathbf{Q}}_p)$ varies over arithmetic primes of a fixed weight $k \geq 2$, we also construct an analogous measure interpolating the $p$-adic Abel–Jacobi image of generalised Heegner cycles associated with the corresponding specialisations of the convolution. The details will appear elsewhere.

**Acknowledgements**
We are grateful to our advisor Haruzo Hida for continuous guidance and encouragement. The topic essentially arose from conversations with him. We thank Francesc Castella, Ming-Lun Hsieh and Chandrashekhar Khare for helpful comments and suggestions. Finally, we are indebted to the referee. The current form of the article owes much to the suggestions of the referee. Particularly, the context of the variation of arithmetic invariants and Sect. 6 arose from the suggestions.

Received: 12 May 2015 Accepted: 17 February 2016
Published online: 01 July 2016

**References**

1. Bertolini, M., Darmon, H., Prasanna, K.: Generalised Heegner cycles and $p$-adic Rankin L-series. Duke Math. J. **162**(6), 1033–1148 (2013)
2. Bertolini, M., Darmon, H., Prasanna, K.: $p$-adic L-functions and the coniveau filtration on chow groups. Journal fur die Reine und Angewandte Mathematik. "http://www.math.mcgill.ca/darmon/pub/pub.html"
3. Burungale, A.: On the non-triviality of the $p$-adic Abel–Jacobi image of generalised Heegner cycles modulo $p$, I: modular curves, preprint, 2014. "http://arxiv.org/abs/1410.0300"
4. Finis, T.: Divisibility of anticyclotomic L-functions and theta functions with complex multiplication. Ann. Math. (2) **163**(3), 767–807 (2006)
5. Hida, H.: On $p$-adic L-functions of GL(2) × GL(2) over totally real fields. Ann. Inst. Fourier(Grenoble) **41**(2), 311–391 (1991)
6. Hida, H., Tilouine, J.: Anticyclotomic Katz $p$-adic L-functions and congruence modules. Ann. Sci. Ecole Norm. Sup. (4) **26**(2), 189–259 (1993)

7. Hida, H.: Non-vanishing modulo $p$ of Hecke L-values. In: Adolphson, A., Baldassarri, F., Berthelot, P., Katz, N., Loeser, F. (eds.) Geometric Aspects of Dwork Theory, pp. 735–784. Walter de Gruyter, Berlin (2004)
8. Hida, H.: p-Adic Automorphic Forms on Shimura Varieties. Springer Monographs in Mathematics. Springer, New York (2004)
9. Hida, H.: Hilbert Modular Forms and Iwasawa Theory. Oxford Mathematical Monographs. Oxford University Press, Oxford (2006)
10. Hida, H.: Non-vanishing modulo $p$ of Hecke L-values and applications. Lond. Math. Soc. Lect. Note, Ser. **320**, 207–269 (2007)
11. Hida, H.: Irreducibility of the Igusa tower. Acta Math. Sin. (Engl. Ser.) **25**, 1–20 (2009)
12. Hida, H.: The Iwasawa $\mu$-invariant of p-adic Hecke L-functions. Ann. Math. (2) **172**, 41–137 (2010)
13. Hida, H.: Elliptic Curves and Arithmetic Invariants. Springer Monographs in Mathematics. Springer, New York (2013). xviii+449 pp
14. Hida, H.: Image of $\Lambda$-adic Galois representation modulo $p$. Invent. Math. **194**(1), 1–40 (2013)
15. Hida, H.: Transcendence of Hecke operators in the big Hecke algebra. Duke Math. J. **163**(9), 1655–1681 (2014)
16. Hsieh, M.-L.: On the non-vanishing of Hecke L-values modulo $p$. Am. J. Math. **134**(6), 1503–1539 (2012)
17. Hsieh, M.-L.: On the $\mu$-invariant of anticyclotomic $p$-adic L-functions for CM fields. J. Reine Angew. Math. **688**, 67–100 (2014)
18. Hsieh, M.-L.: Special values of anticyclotomic Rankin–Selbeg L-functions. Doc. Math. **19**, 709–767 (2014)
19. Katz, N.M.: $p$-Adic L-functions for CM fields. Invent. Math **49**(3), 199–297 (1978)
20. Kim, D.: On the transfer congruence between $p$-adic Hecke L-functions, preprint, (2014). http://arxiv.org/abs/1411.1184
21. Shimura, G.: On analytic families of polarized abelian varieties and automorphic functions. Ann. Math. (2) **78**, 149–192 (1963)
22. Shimura, G.: Abelian Varieties with Complex Multiplication and Modular Functions. Princeton Mathematical Series, vol. 46. Princeton University Press, Princeton (1998)
23. Sinnott, W.: On a theorem of L. Washington, Journées arithmétiques de Besançon (Besançon, 1985). Astérisque No. 147–148 (1987), 209–224, 344
24. Sun, H.-S.: Homological interpretation of a theorem of L. Washington. J. Number Theory **127**(1), 47–63 (2007)
25. Vatsal, V.: Special values of anticyclotomic L-functions. Duke Math. J. **116**(2), 219–261 (2003)
26. Washington, L.: The non-$p$-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension. Invent. Math. **49**(1), 87–97 (1978)